Problem Set 8 – Due Thursday, November 29, 2012

This is a crucial problem set; with it, you should be coming to understand one of the most important concepts in this class: reductions.

With this problem set, you are back to doing problem sets on your own.

Problem 1.[1] Classify each of the following languages as either **recursive**, or **r.e.** but not not co-r.e., or **co-r.e.** but not r.e., or **neither** r.e. nor co-r.e. (For ease of grading, please use these four labels.) You should be able to prove all of your claims, but, to keep things short, please provide a proof only for problems marked with a star. Proofs that a language is not r.e. or not co-r.e. must take the form of a reduction.

A $\{\langle M\rangle : M$ is a TM that accepts some string of prime length$\}$.

B$\star$ $\{\langle M, k\rangle : M$ is a TM that accepts at least one string of length $k\}$.

C $\{\langle M\rangle : M$ is a TM and M has 100 states$\}$.

D $\{\langle M\rangle : M$ is a TM and $L(M) = L(M)^*\}$.

E $\{\langle M\rangle : M$ is a TM and $L(M) = \emptyset\}$.

F $\{\langle M\rangle : M$ is a TM and $L(M)$ is r.e.$\}$.

G$\star$ $\{\langle M, k\rangle : M$ is a TM that runs forever (loops) on at least one string of length $k\}$.

H $\{\langle M\rangle : M$ is a C-program that halts on $\langle M\rangle\}$.

I$\star$ $\{\langle M, k\rangle : M$ is a TM that accepts a string of length k and diverges on a string of length $k\}$. Assume that the underlying alphabet has at least two characters.

J $\{\langle M\rangle : M$ is a TM and M will visit state q_{20} when run on some input $x\}$.

K$\star$ $\{\langle M, w\rangle : M$ is a TM and M that uses at most 20 tape cells when run on $w\}$.

L $\{\langle G\rangle : G$ is a CFG and G accepts an odd-length string$\}$.

M $\{\langle G_1, G_2\rangle : G_1$ and G_2 are CFGs and $L(G_1) = L(G_2)\}$.

N$\star$ $\{\langle M\rangle :$ M is a TM that accepts some palindrome$\}$.

Problem 2 Say that a language $L = \{x_1, x_2, \ldots\}$ is *enumerable* if there exists a two-tape TM M that outputs $x_1\sharp x_2\sharp x_3\sharp\cdots$ on a designated *output tape.* The other tape is a designated *work tape*, and the output tape is write-only, with the head moving only from left-to-right. Say that L is *enumerable in lexicographic order* if L is enumerable, as above, and, additionally, $x_1 < x_2 < x_3 < \cdots$, where "<" denotes the usual lexicographic ordering on strings.

[1] Will count as more than one problem.

A. Prove that L is r.e. iff L is enumerable. (This explains the name "recursively enumerable.")

B. Prove that L is recursive iff it is enumerable in lexicographic order.

Problem 3 Prove or disprove each of the following claims.

A. $A \leq_{\mathrm{m}} A$.

B. If $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} C$, then $A \leq_{\mathrm{m}} C$.

C. If $A \leq_{\mathrm{m}} B$ then $\overline{A} \leq_{\mathrm{m}} \overline{B}$.

D. If A is r.e. and $A \leq_{\mathrm{m}} \overline{A}$ then A is recursive.

E. If A is recursive, then $A \leq_{\mathrm{m}} a^*b^*$.

F. If $A \leq_{\mathrm{m}} B$ then $B \leq_{\mathrm{m}} A$.

G. If $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} A$ then $A = B$.