Problem Set 8 – Due Friday, May 24, 2013

If you liked working with a partner and want to do so again, you may, turning in one problem set per group. I don't recommend groups of more than two, but if you have a good partnership going with three, I won't complain.

Problem 1.[1] Classify each of the following languages as either **recursive**, or **r.e.** but not not co-r.e., or **co-r.e.** but not r.e., or **neither** r.e. nor co-r.e. (For ease of grading, please use these four labels.) Provide a proof for all classification claims. Proofs that a language is not r.e. or not co-r.e. must take the form of a many-one reduction.

A $\{\langle M, k\rangle : M$ is a TM that accepts at least one string of length $k\}$.

B $\{\langle M, k\rangle : M$ is a TM that runs forever (loops) on at least one string of length $k\}$.

C $\{\langle M, k\rangle : M$ is a TM that accepts a string of length k and diverges on a string of length $k\}$. Assume that the underlying alphabet has at least two characters.

D $\{\langle M, w\rangle : M$ is a TM and M that uses at most 20 tape cells when run on $w\}$.

E $\{\langle M\rangle :$ M is a TM that accepts some palindrome$\}$.

Problem 2 Say that a language $L = \{x_1, x_2, \ldots\}$ is *enumerable* if there exists a two-tape TM M that outputs $x_1\sharp x_2\sharp x_3\sharp\cdots$ on a designated *output tape*. The other tape is a designated *work tape*, and the output tape is write-only, with the head moving only from left-to-right. Say that L is *enumerable in lexicographic order* if L is enumerable, as above, and, additionally, $x_1 < x_2 < x_3 < \cdots$, where "$<$" denotes the usual lexicographic ordering on strings.

A. Prove that L is r.e. iff L is enumerable. (This explains the name "recursively enumerable.")

B. Prove that L is recursive iff it is enumerable in lexicographic order.

Problem 3 Prove or disprove each of the following claims.

A. $A \le_{\rm m} A$.

B. If $A \le_{\rm m} B$ and $B \le_{\rm m} C$, then $A \le_{\rm m} C$.

C. If $A \le_{\rm m} B$ then $\overline{A} \le_{\rm m} \overline{B}$.

D. If A is r.e. and $A \le_{\rm m} \overline{A}$ then A is recursive.

E. If A is recursive, then $A \le_{\rm m} a^*b^*$.

F. If $A \le_{\rm m} B$ then $B \le_{\rm m} A$.

G. If $A \le_{\rm m} B$ and $B \le_{\rm m} A$ then $A = B$.

Problem 4. Let us say that a nonempty set B is *countable* if you can list (possibly with repetitions) its elements $B = \{a_1, a_2, a_3, \ldots\}$; more formally, there is a surjective[2] function f from $\mathbb{N}$ to B. We'll say that the empty set is also countable. An set is *uncountable* if it is not countable.

A. Prove that any subset A of a countable set B is countable.

B. Fix an alphabet Σ. Prove that there are countably many finite languages over Σ.

C. Fix an alphabet Σ. Prove that there are uncountably many infinite languages over Σ.

[1] A particularly important problem, to be able to do reductions like these.

[2] Recall that a function $f : A \to B$ is surjective (or onto) if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.