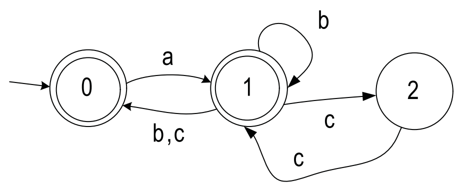
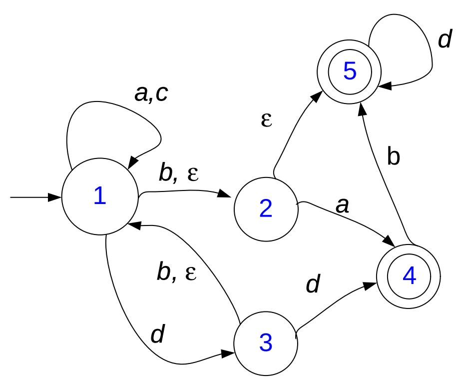
Problem Set 3 Solutions

Problem 1. *Using the procedure shown in class, convert the following NFA into a DFA for the same language.*



The problem is pretty mechanical—I'm not going to draw out the solution—hopefully you didn't have trouble doing so.

Problem 2. *Using the procedure shown in class, eliminate all ε-arrows from the following NFA.*



The problem too is mechanical. States 1, 2, and 3 all become final (so all states are now final), since they can reach final states along ε-paths. Now we add in "bypass arcs." The approach I explained in class for doing this: for each state p of the NFA, in parallel: find all all states q reachable from p along ε-paths; find each transition to a state r labeled by a character $a \in \Sigma$; add in a direct connection, if needed, from p to r labeled by a. After all this is done, eliminate all ε-transitions.

Problem 3. *Let $L_1, L_2, L_3 \subseteq \Sigma^*$ be languages and let $Most(L_1, L_2, L_3)$ be the set of all $x \in \Sigma^*$ that are in at least two of L_1, L_2, L_3. Prove: if L_1, L_2, and L_3 are DFA-acceptable then so is $Most(L_1, L_2, L_3)$.*

Solution 1: Extend the product construction. Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, and $M_3 = (Q_3, \Sigma, \delta_3, q_3, F_3)$ be DFAs for L_1, L_2, and L_3, respectively. Form a new DFA $M = (Q, \Sigma, \delta, s, F)$ for $Most(L_1, L_2, L_3)$ be defining $Q = Q_1 \times Q_2 \times Q_3$, $s = (q_1, q_2, q_3)$, $\delta((p, q, r), a) = (\delta_1(p, a), \delta_2(q, a), \delta_3(r, a))$, and $F = \{(p, q, r) \in Q_1 \times Q_2 \times Q_3$: at least two of the following three things are true: $p \in F_1$, $q \in F_2$, $r \in F_3\}$. It is easy to see that $L(M) = Most(L_1, L_2, L_3)$.

Solution 2: Use closure properties. Note that $Most(L_1, L_2, L_3) = (L_1 \cap L_2) \cup (L_2 \cap L_3) \cup (L_1 \cap L_3)$. The regular languages are closed under $\cap$ and $\cup$ and so they are closed under *Most*.

Problem 4 *Let $Stutter(L) = \{a_1a_1\ a_2a_2\ \cdots\ a_na_n \in \Sigma^* : a_1a_2 \cdots a_n \in L\}$.* ***(A)*** *Prove that the DFA-acceptable languages are closed under Stutter.* ***(B)*** *Then, having proved it once, give another, entirely different proof.*

Here three different proofs:

(1) Consider the map $h\colon \Sigma \to \Sigma^*$ defined by $h(a) = aa$ for all $a \in \Sigma$. Then $\mathit{Stutter}(L) = h(L)$. We know that the DFA/NFA-acceptable languages are closed under homomorphism (from a previous problem set), so we are done.

(2) Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting L. To make an NFA accepting $\mathit{Stutter}(L)$, add a state "in the middle of each arrow" to ensure that a symbol $a \in \Sigma$ is always followed by a symbol a, and the same destination is then reached. This would give an NFA for $\mathit{Stutter}(L)$. You could, if desired, make it into a DFA by the addition of a dead state that was connected up to the rest of the machine in the natural way.

(3) Use the regular-expression characterization of the DFA-acceptable languages. Let α be a regular expression over Σ. Construct from α a new regular expression β by replacing each character $a \in \Sigma$ that occurs in α by $(a \circ a)$. What results is a new regular expression β where $L(\beta) = \mathit{Stutter}(L(\alpha))$.

Problem 5. *How many states are in the smallest possible DFA for $\{0,1\}^*\{1^{10}\}$? Prove your result.*

First, 11 states are *sufficient*: there is a DFA M_{11} that accepts $L = \{0,1\}^*\{1^{10}\}$ and has 11 states. The machine has states $Q = \{q_0, q_1, \ldots, q_{10}\}$ with q_0 the start state, $F = \{q_{10}\}$ the final states, $\delta(q, 0) = q_0$ for all states $q \in Q$, while $\delta(q_i, 1) = q_{i+1}$ for $i < 10$ and $\delta(q_{10}, 1) = q_{10}$.

Second, 11 states are *necessary*. Suppose for contradiction that there exists a 10-state DFA $M = (Q, \Sigma, \delta, q_0, F)$ that accepts L. Consider the 11 strings 1^i for $0 \le i \le 10$. By the pigeonhole principle we know that $\delta^*(q_0, 1^i) = \delta^*(q_0, 1^I)$ for $0 \le i < I \le 10$. But then $\delta^*(q_0, 1^i 1^{10-I}) = \delta^*(q_0, 1^I 1^{10-I})$, so $\delta^*(q_0, 1^{10-j}) = \delta^*(q_0, 1^{10})$ for some $j \ge 1$. But the lefthand state must be outside F and the righthand states must be in F, a contradiction.

One could use the DFA minimization procedure to prove this, establishing that M_{11} is already a minimal-size DFA. Here one shows that no two states are *equivalent*, which follows, we have claimed, by showing that the algorithm of class discovers no *in*equivalence when looking at 0-and 1-character extensions.

Problem 6 *Let L_n (for $n \ge 1$) be $\{0,1\}^*\{1\}\{0,1\}^n$. Prove that there is an NFA for L_n having $n+2$ states, but that there is no DFA for L_n having $2^n - 1$ or fewer states. In a well written English sentence or two, give a high-level interpretation of your result.*

As with the last problem, the first part is constructive; just draw the needed machine. For the second part, assume for contradiction that there is a $(2^n - 1)$-state DFA $M = (Q, \Sigma, \delta, q_0, F)$. By the pigeonhole principle, we know that some two distinct strings $x, x' \in \{0,1\}^n$ satisfy $\delta(q_0, x) = \delta(q_0, x')$. Since x and x' differ, they do so at some particular bit position $\ell \in [1..n]$ (numbering from 1, starting on the left). Let x_0 be the one of x, x' with $x_0[\ell] = 0$ and let x_1 be the one of x, x' with $x_1[\ell] = 1$. Now consider the strings $y_0 = x_0\, 0^\ell$ and $y_1 = x_1\, 0^\ell$. The second is in L_n; the first is not. But we know that $\delta^*(q_0, y_0) = \delta^*(q_0, y_1)$, getting us our contradiction: this state cannot be both final and nonfinal.

Interpretation of the result: *There can be an exponential gap between the size of the smallest NFA for a language and the size of the smallest DFA for it.* Or, said differently, *Some languages can be represented* ***much*** *more efficiently with an NFA than a DFA.*