Problem Set 7 – Due Tuesday, March 6, 2012

Warning: this is a long but important homework.

Problem 1. Classify each of the following languages as either (a) **recursive**—I see how to decide this language; (b) **r.e.**—I don't see how to decide this language, but I can see a procedure to accept this language; (c) **co-r.e.**—I don't see how to decide this language, but I can see a procedure to accept the complement of the language; or (d) **neither**: I don't see how to accept this language nor its complement. No justification is needed for your answers.

Part A. $\{\langle M\rangle : M$ is a TM that accepts some string of prime length$\}$.

Part B. $\{\langle M\rangle : M$ is a TM and M has 100 states$\}$.

Part C. $\{\langle M\rangle : M$ is a TM and $L(M) = L(M)^*\}$.

Part D. $\{\langle M\rangle : M$ is a TM and $L(M)$ is r.e. $\}$.

Part E. $\{\langle M\rangle : M$ is a C-program that halts on $\langle M\rangle\}$.

Part F. $\{\langle M\rangle : M$ is a TM and M will visit state q_{20} when run on some input $x\}$.

Part G. $\{\langle M\rangle : M$ is a TM and M that uses at most 20 tape cells when run on blank tape$\}$.

Part H. $\{\langle G\rangle : G$ is a CFG and G accepts an odd-length string$\}$.

Part I. $\{\langle G_1, G_2\rangle : G_1$ and G_2 are CFGs and $L(G_1) = L(G_2)\}$.

Problem 2. Prove whether each of the following languages is **recursive**, **r.e.** but not recursive, **co-r.e.** but not recursive, or **neither** r.e. nor co-r.e.

Part A. $L = \{\langle M, w\rangle : M$ is a TM that uses at most 20 tape squares when run on $w\}$.

Part B. $L = \{\langle M, k\rangle : M$ is a TM that accepts at least one string of length $k\}$.

Part C. $L = \{\langle M, k\rangle : M$ is a TM that diverges (loops) on at least one string of length $k\}$.

Part D. $L = \{\langle M, k\rangle : M$ is a TM that accepts a string of length k and diverges on a string of length $k\}$. Assume that the underlying alphabet has at least two characters.

Part E. $L = \{\langle M\rangle :\ M$ is a TM that accepts some palindrome$\}$.

Problem 3 Say that a language $L = \{x_1, x_2, \ldots\}$ is *enumerable* if there exists a two-tape TM M that outputs $x_1\sharp x_2\sharp x_3\sharp\cdots$ on a designated *output tape*. The other tape is a designated *work tape*, and the output tape is write-only, with the head moving only from left-to-right. Say that L is *enumerable in lexicographic order* if L is enumerable, as above, and, additionally, $x_1 < x_2 < x_3 < \cdots$, where "$<$" denotes the usual lexicographic ordering on strings.

Part A. Prove that L is r.e. iff L is enumerable. (This explains the name "recursively enumerable.")

Part B. Prove that L is recursive iff it is enumerable in lexicographic order.

Problem 4* *Challenging.* An *unrestricted grammar* $G = (V, \Sigma, R, S)$ is like a CFG except that rules have lefthand sides from $(\Sigma \cup V)^*V(\Sigma \cup V)^*$. Whenever you have a rule $\alpha \to \beta$, you can replace α, wherever it occurs in a sentential form σ within a derivation, with β. The language of an unrestricted grammar G is, as usual, the set of terminal strings derivable from the start symbol: $L(G) = \{x \in \Sigma^* :\ S \stackrel{*}{\Rightarrow} x\}$. Show that the languages of unrestricted grammars are exactly the r.e. languages.