Problem Set 8 – Due Wednesday, 4:15 pm, November 27, 2013

1. Let us say that the *weight* of a Thanksgiving turkey is the amount it weighed on the midnight before its death, rounded to the nearest milligram. Prove: *on Thanksgiving day, two of the consumed turkeys will have the same weight.* Describe any assumption you need to make and why they are reasonable.

2. Seven boys and five girls are seated (in an equally spaced fashion) around a circular table with 12 chairs. Prove that there are two boys sitting opposite one another.

3. Prove: if you select 27 distinct odd numbers between 1 and 100, some two of them will sum to 102.

4. Select 11 different number from $\{1, 2, ..., 20\}$. Prove that two of your numbers, a and b, will differ by two.

5. Draw a graph on five vertices that contains no clique of size three (that is, no triangle) and no anti-clique of size three (that is, three vertices none of which is connected to any other).

6. Let us call graphs $G = (V, E)$ and $G' = (V', E')$ *fundamentally different* if they are not isomorphic. How many fundamentally different graphs are there on four vertices?

7. A *call graph* on US-phone metadata is a graph $G = (V, E)$ that has a vertex $v \in V$ for each US phone number and an edge $\{v, w\} \in E$ if there has ever been a phone call between v and w. The NSA allows analysts to investigate the calls of a set $S \subseteq V$ of court-authorized *seeds*. It also allows investigation of vertices at a distance three or less from a seed (that is, three or fewer *hops* away). Assume $|S| = 300$ and each phone has had contact with exactly 100 others. Under these assumptions, give a tight upperbound on the number of phone numbers about which warrantless queries are permitted.

8. The following student answers appeared on Quiz #3 as the first part of Problem 2. Classify each solution as:

   - **Right.** A correct definition.
   - **Wrong.** A mathematically clear statement, but not a correct definition. In this case, translate the statement into first-order logic.
   - **Garbled.** Not precise enough to have a clear meaning. This includes both agrammatical English and things that look like math but aren't precise.

   Explain all answers for which an explanation seems beneficial (for correct answers and things that are utterly garbled, there might be nothing much one can say).

   Keep in mind that students are completing a sentence. That sentence begins:

   A function $f : A \to B$ is **one-to-one** if

   (a) $f : A \to B\{x : x \in A, \exists x \in B\}$
   (b) each value in the domain maps to **one** value in the codomain
   (c) elements in A are pair with a unique element in B
   (d) $(\forall a)(\forall b)(f(a) = f(b) \to b)$ $(a, b \in A)$
   (e) $(\forall a)(\forall a')(a \neq a' \to f(a) \neq f(a'))$

(f) no two values in domain have the same value in the codomain

(g) a domain A is mapped to one and only one codomain B

(h) for every element in B, only one preimage exists in A

(i) all the elements in A have connected to elements in B

(j) $|A| \leq |B|$

(k) $f(x) = f(y)$ and $x = y$ where $x, y \in A$

(l) no two points in the domain map to the same image under f