Shape Analysis with the Delaunay Triangulation

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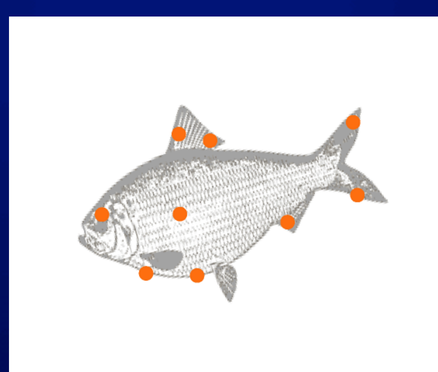
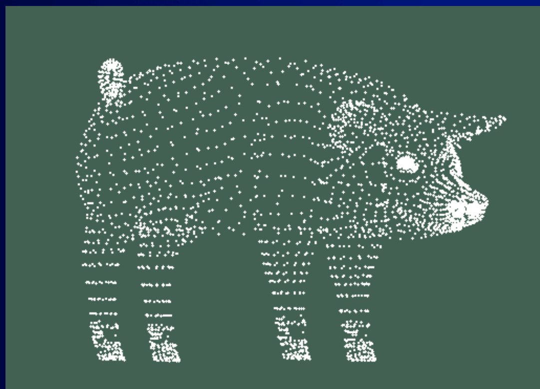
Shape of a Point Set



Surface Reconstruction



Input: *Samples* from object surface.

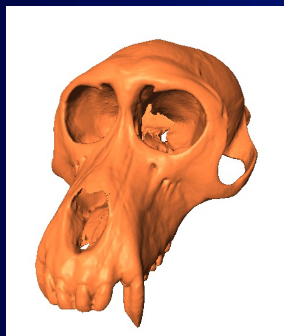
Output: Polygonal model.

Point Set Capture

Cyberware model 15

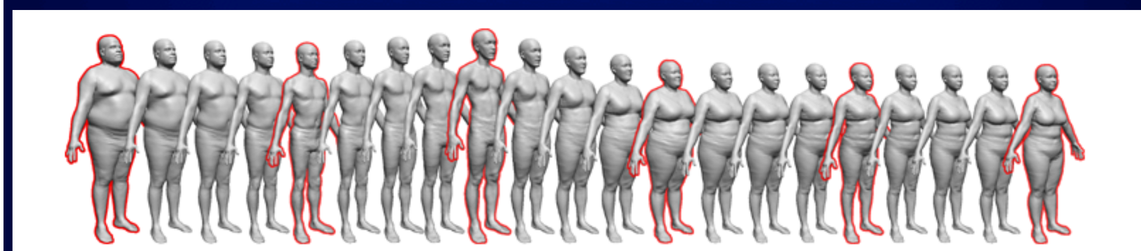
Point Grey Bumblebee

Applications



Delson et al, AMNH

Levoy et al, Stanford

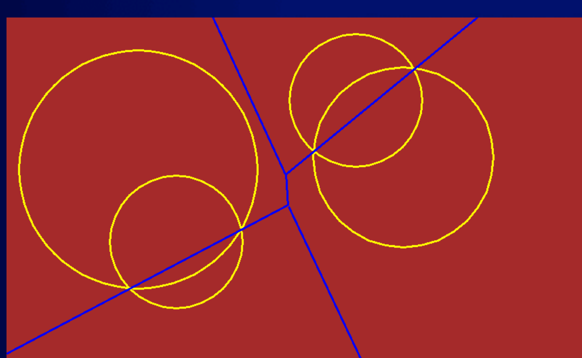


Allen, Curless, Popovic, U Wash.

Power Diagram

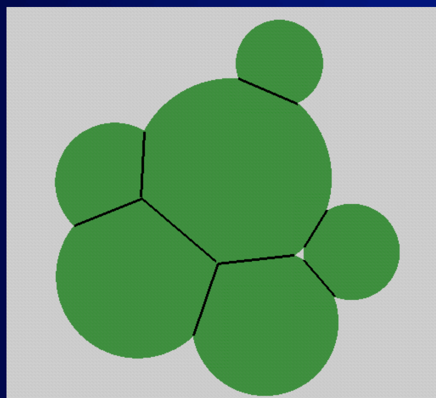
Weighted Voronoi diagram. Input: balls.

Dist(x, ball) = $\text{dist}^2(x,\text{center}) - \text{radius}^2$



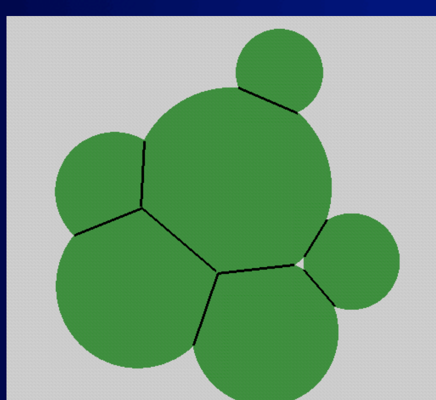
Polyhedral cells, same algorithm as regular Voronoi diagram (lift to convex hull)

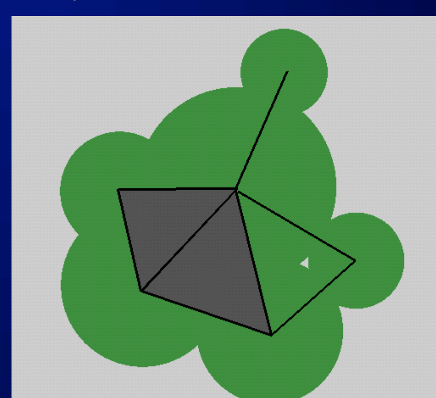
Alpha-shapes



Overlay Voronoi diagram of balls on union of input balls. Discard features outside of the union.

Alpha-shapes





Alpha shape is the set of weighted Delaunay features dual to the weighted Voronoi features intersecting union of balls.

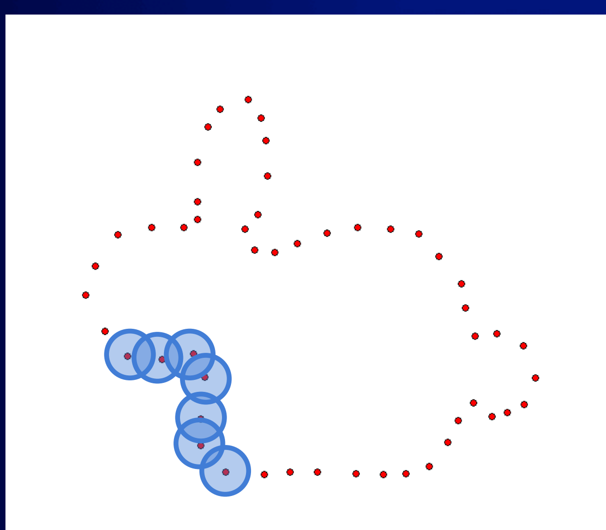
Alpha-shapes

Edelsbrunner, Kirkpatrick, Seidel, 83

Edelsbrunner, 93: Alpha shape is homotopy equivalent to union of balls, close correspondence with union structure.

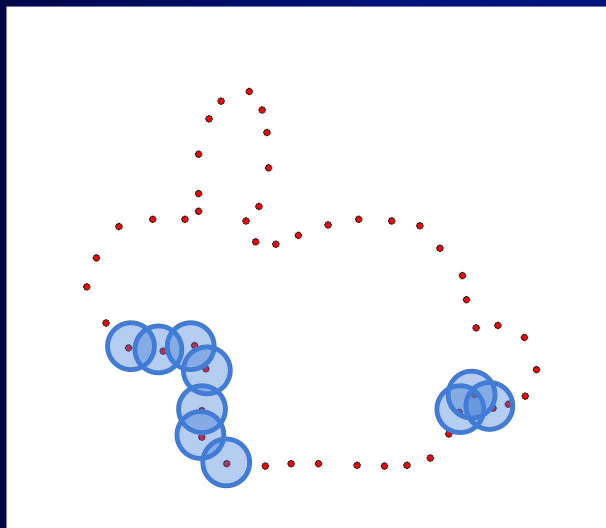
Edelsbrunner & Muecke, 94: 3D surface reconstruction.

Alpha-shape reconstruction



Put small ball around each sample, compute alpha-shape.

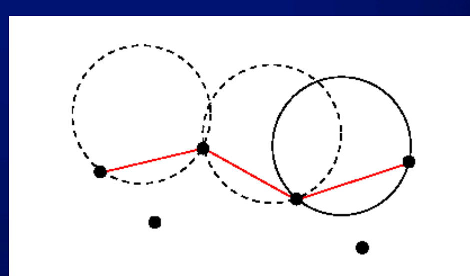
Difficulty



Usually no ideal choice of radius.

Ball-pivoting

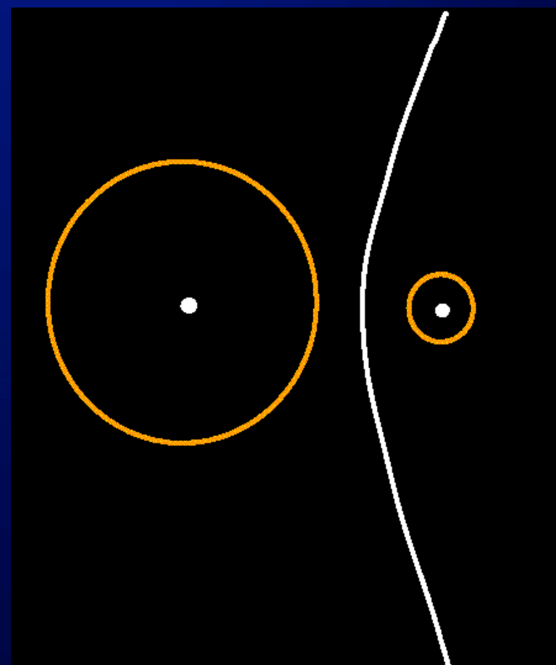
Bernardini et al, IBM



Fixed-radius ball "rolling" over points selects subset of alpha-shape.

Medial Axis

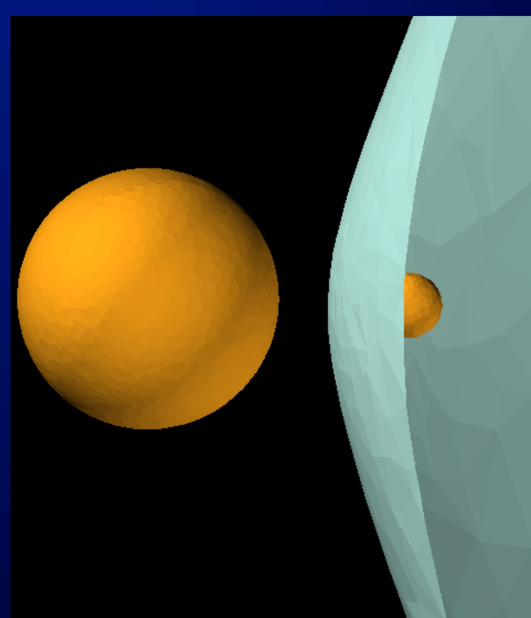
Medial axis is set of points with more than one closest surface point.



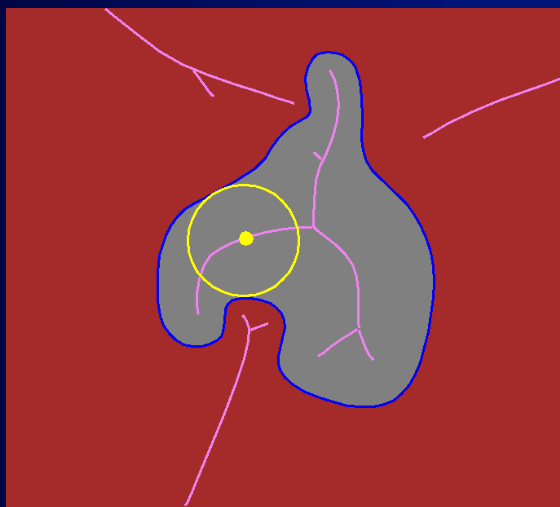
Blum, 67

3D Medial Axis

Medial axis of a surface forms a dual surface.



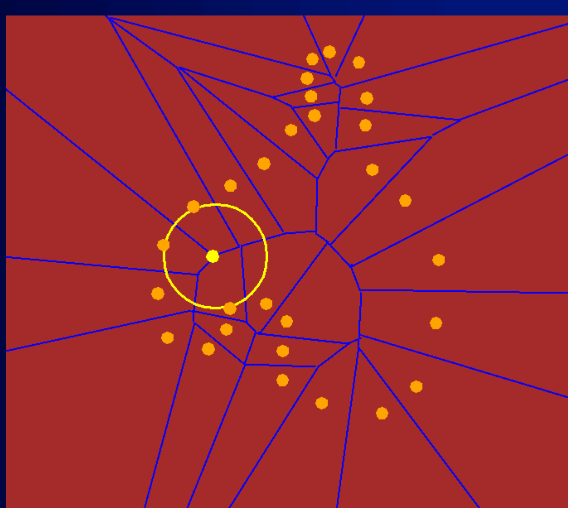
Medial Axis



Maximal ball avoiding surface is a medial ball.

Every solid is a union of balls !

Relation to Voronoi

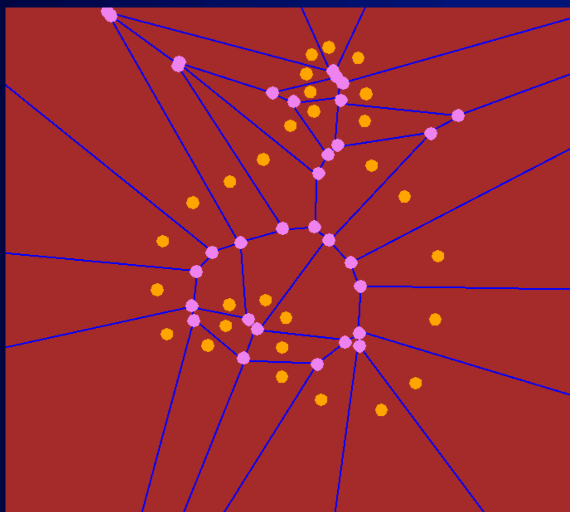


Voronoi balls approximate medial balls.

For dense surface samples in 2D, all Voronoi vertices lie near medial axis.

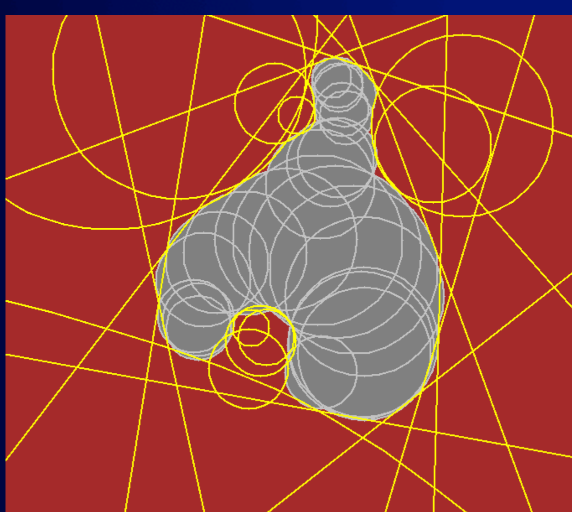
Ogniewicz, 92

Convergence



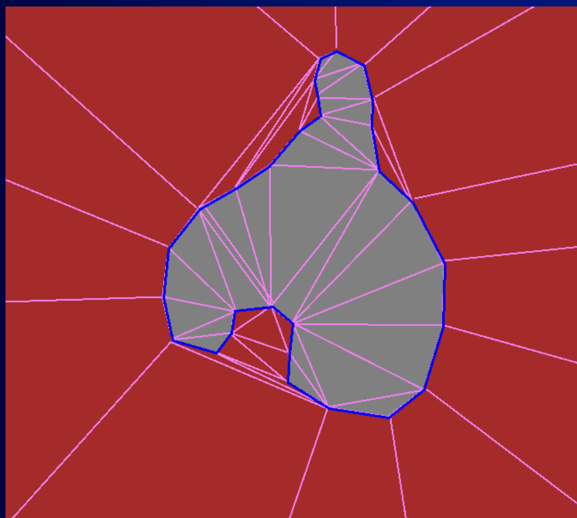
In 2D, set of Voronoi vertices converges to the medial axis as sampling density increases.

Discrete unions of balls



Voronoi balls approximate the object and its complement.

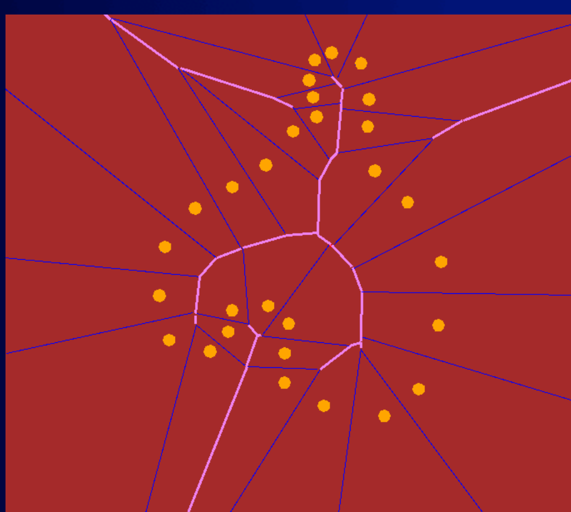
2D Curve Reconstruction



Blue Delaunay edges reconstruct the curve, pink triangulate interior/exterior.

Many algorithms, with proofs, for coloring edges.

2D Medial Reconstruction



Pink approximate medial axis.

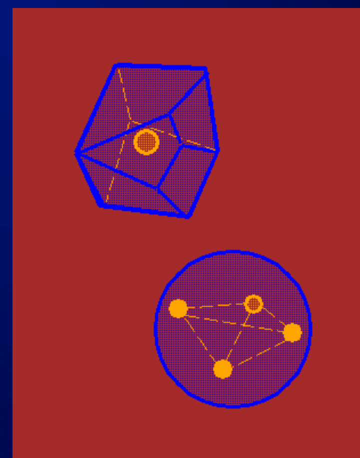
By nerve theorem, approximation is homotopy equivalent to object and its complement.

3D Voronoi/Delaunay

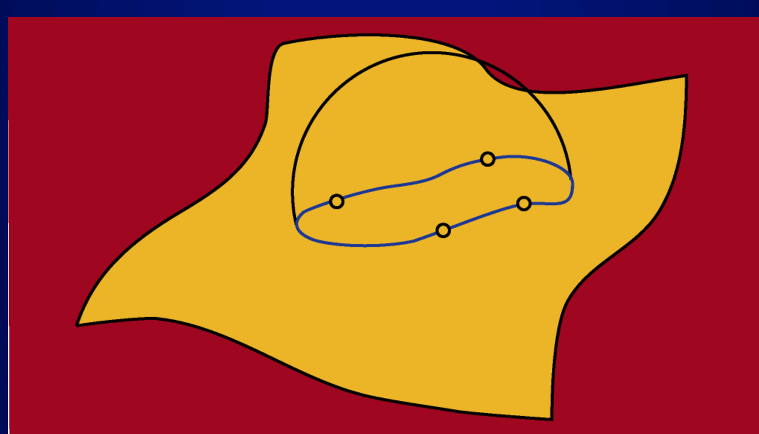
Voronoi cells are convex polyhedra.

Voronoi balls pass through 4 samples.

Delaunay tetrahedra.

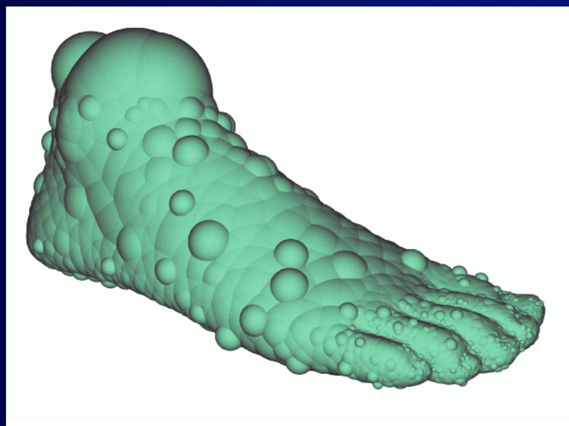


Sliver tetrahedra



In 3D, some Voronoi vertices are not near medial axis …

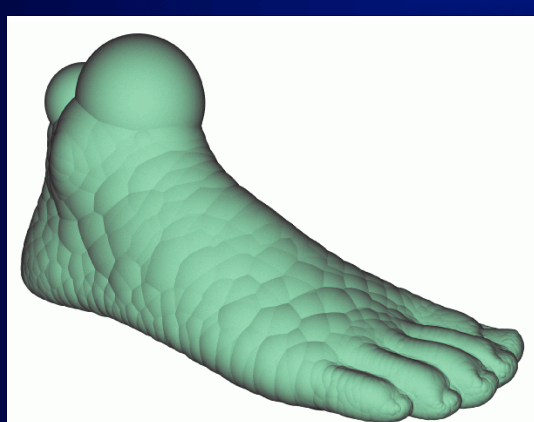
Sliver tetrahedra



Interior Voronoi balls

…. even when samples are arbitrarily dense.

Poles



Interior *polar* balls

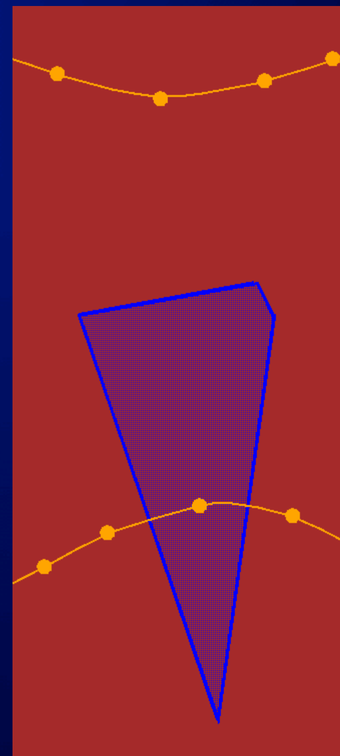
Subset of Voronoi vertices, the poles, approximate medial axis.

Amenta & Bern, 98
"Crust" papers

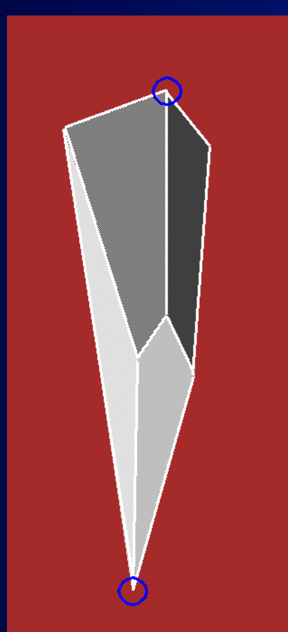
Poles

For dense surface samples, Voronoi cells are:

- long and skinny,
- perpendicular to surface,
- with ends near the medial axis.



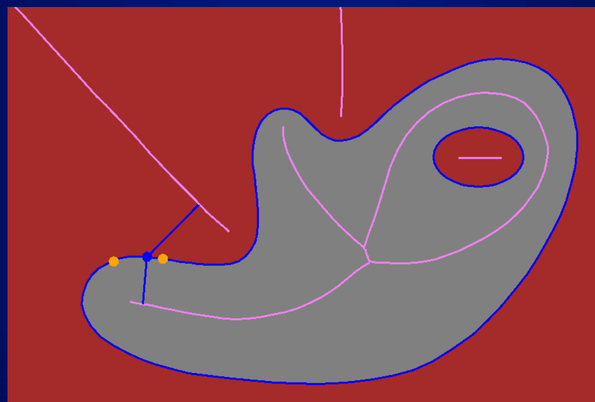
Poles



Poles are Voronoi vertices at opposite ends.

To find: farthest Voronoi vertex from sample, farthest on opposite side.

Sampling Requirement

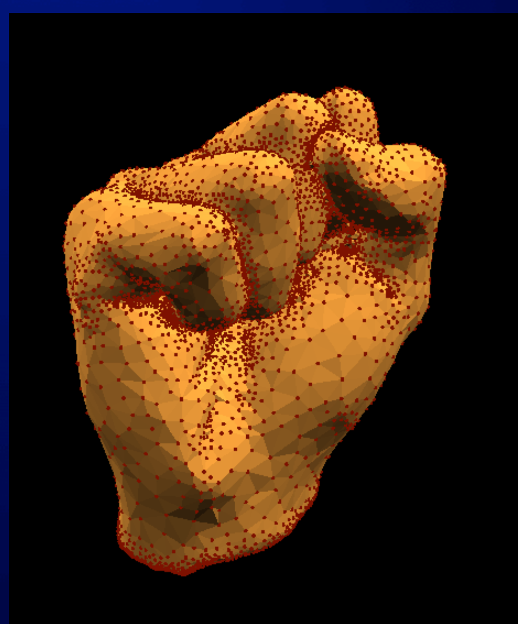


ε-sample: distance from any surface point to nearest sample is at most small constant ε times distance to medial axis.

Note: surface has to be smooth.

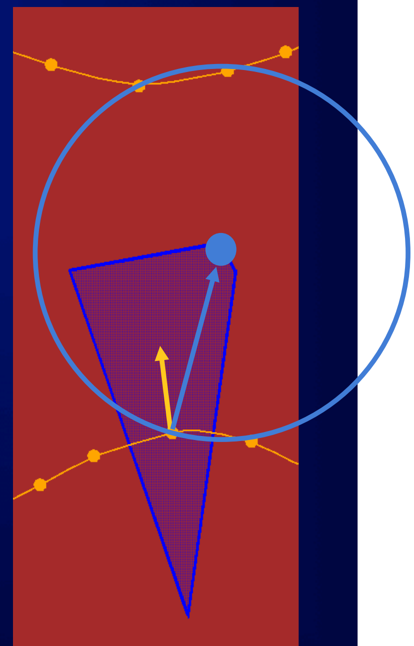
Sampling Requirement

Intuition: dense sampling where curvature is high or near features.



Large balls tangent

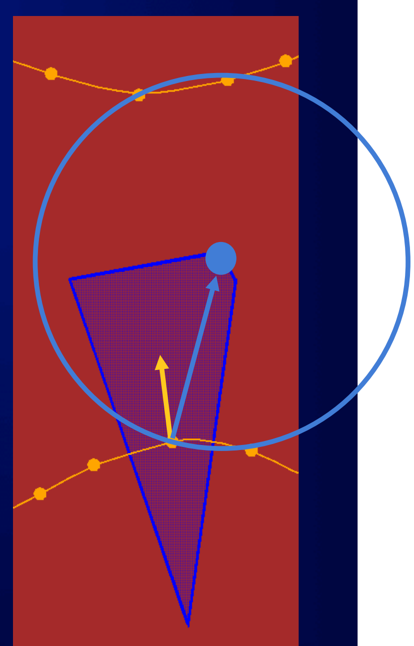
Any large ball (with respect to distance to medial axis) touching sample s has to be nearly tangent to the surface at s.



Specifically

Given an ε-sample from a surface F:

Angle between normal to F at sample s and vector from s to either pole = O (ε)



Results

Look for algorithms where….

Input: ε-sample from surface G

Output: PL-surface,

- near G, converges
- normals near G, converge
- PL manifold
- homeomorphic to G

Formal Algorithms

Amenta and Bern, crust

Amenta, Choi, Dey and Leekha, co-cone

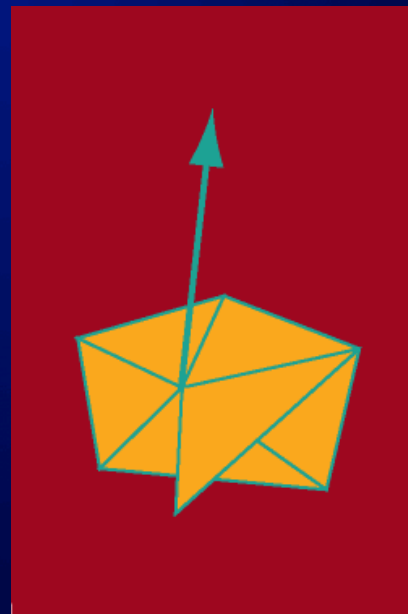
Boissonnat and Cazals, natural neighbor

Amenta, Choi and Kolluri, power crust

Co-cone

Estimate normals, choose candidate triangles with good normals at each vertex.

Extract manifold from candidates.

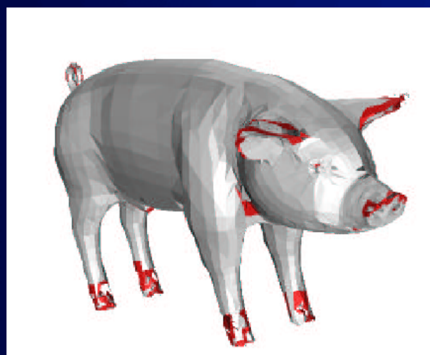


Co-cone

Works well on clean data from a closed surface.

Amenta, Choi, Dey, Leekha 2000

Co-cone extensions

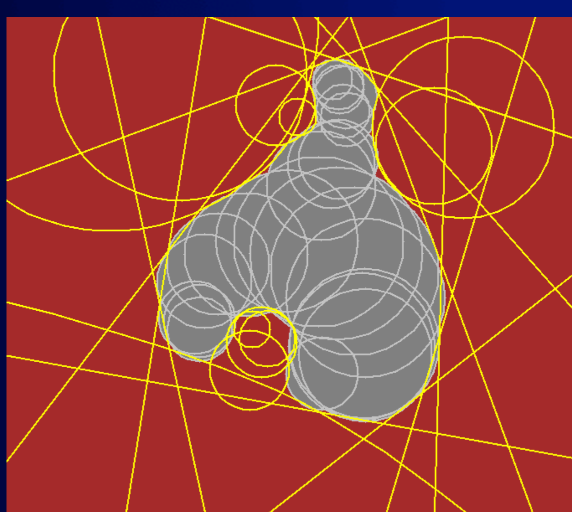


Dey & Giesen, undersampling errors.

Dey & Goswami, hole-filling.

Dey, Giesen & Hudson, divide and conquer for large data.

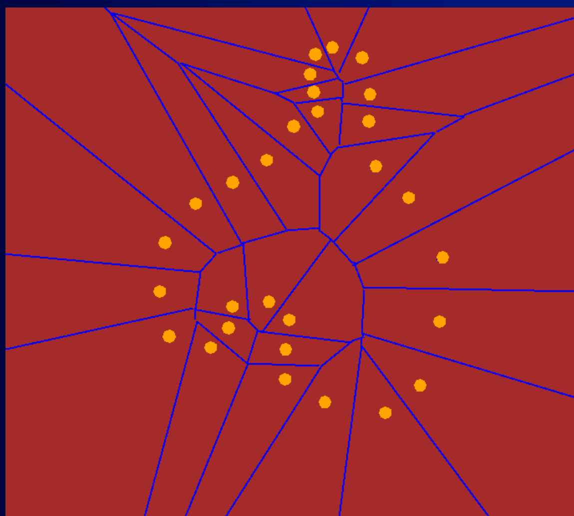
Power Crust



Amenta, Choi and Kolluri, 01

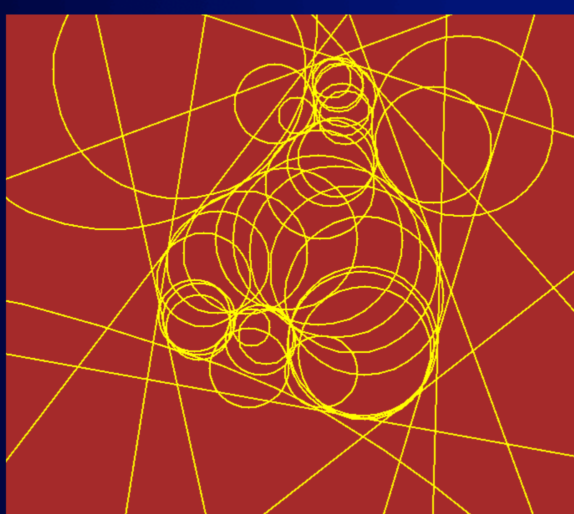
Idea: Approximate object as union of balls, compute polygonal surface from balls.

Power Crust



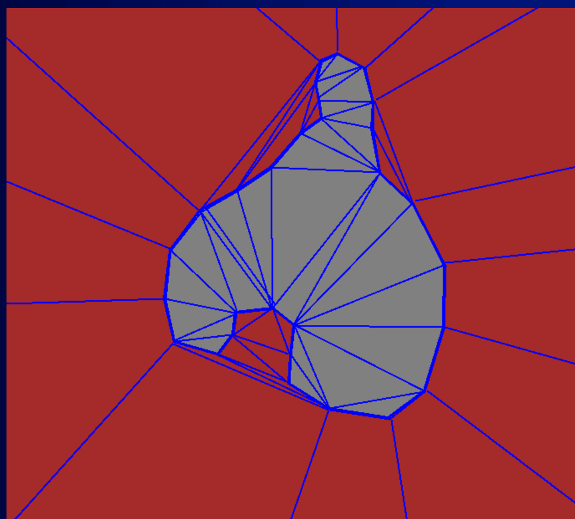
Start with all poles.

Power Crust



Compute polygonal decomposition using power diagram.

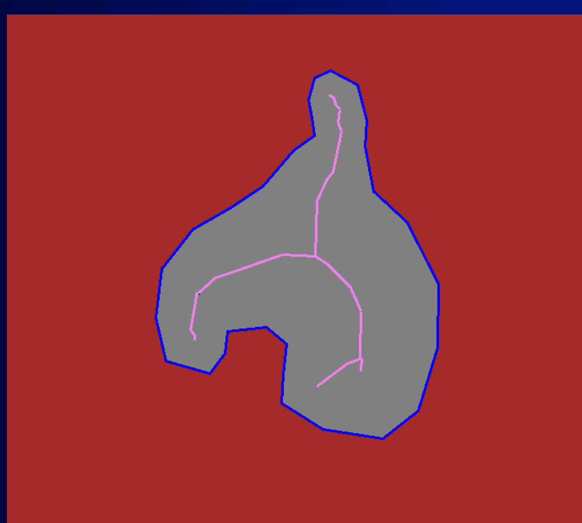
Power Crust



Label power diagram cells inside or outside object (skipping details).

Inside cells form polyhedral solid.

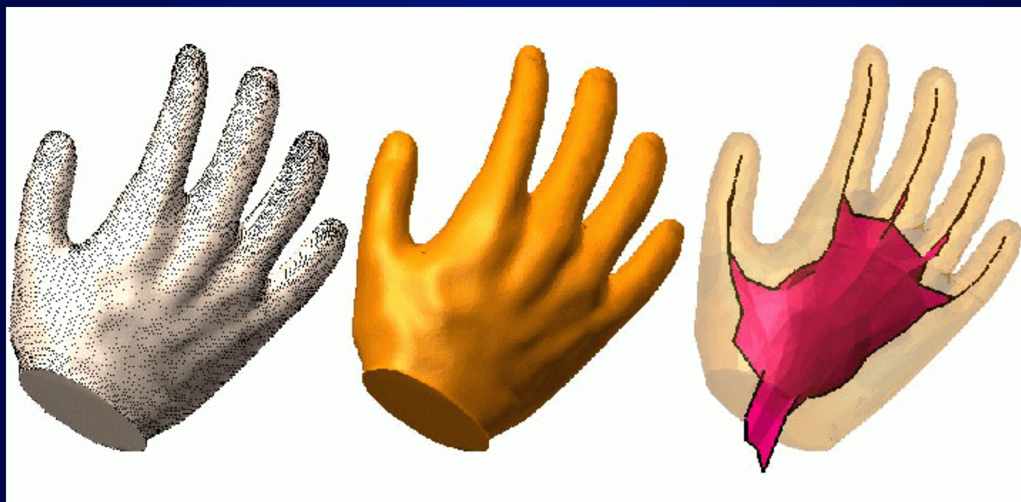
Power Crust



Boundary of solid gives output surface.

Connect inner poles with adjacent power diagram cells for approximate medial axis.

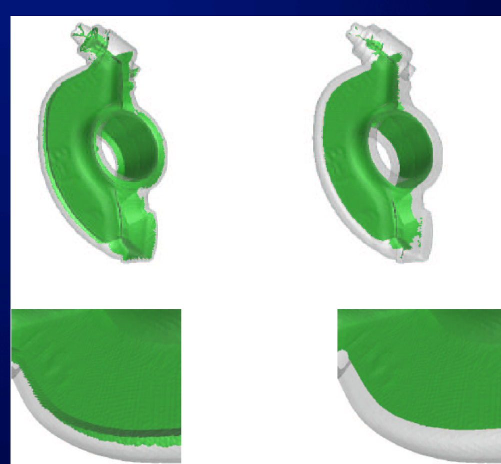
Example



Laser range data, power crust, simplified approximate medial axis.

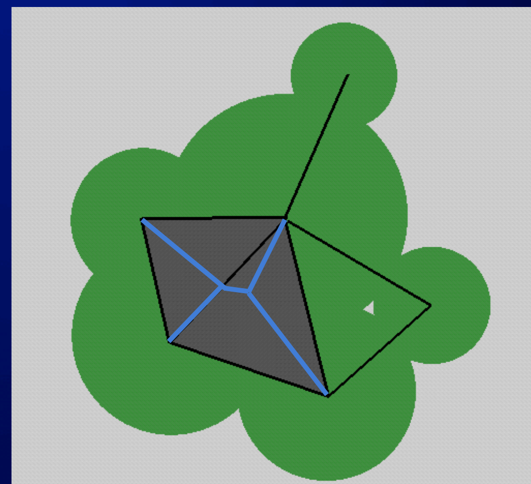
Medial axis approximation

Dey & Zhao, 02
Voronoi diagram
far from surface.



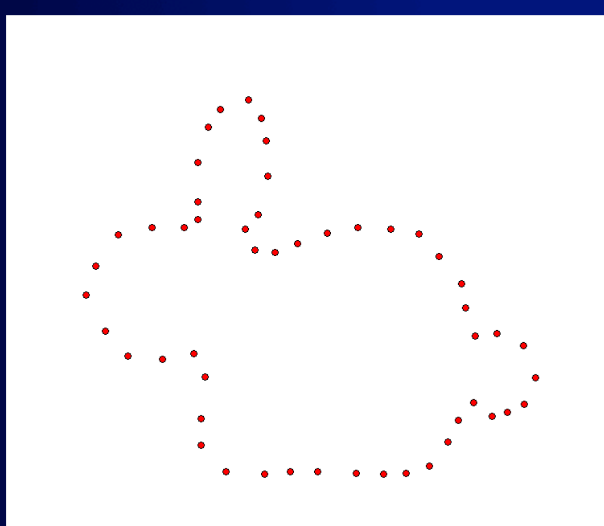
Medial axis approximation

Medial axis of union of balls = lower dimensional parts of alpha shape + intersection with Voronoi diagram of union vertices.



Attali & Montanvert, 97, A & Kolluri, 01

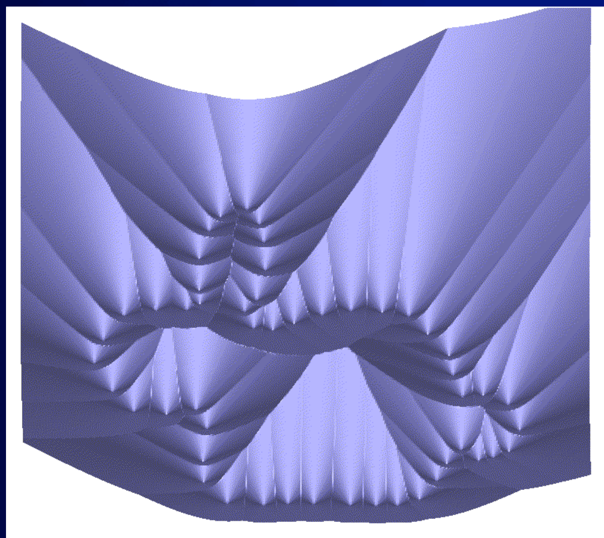
Distance function



Giesen and John, 01,02

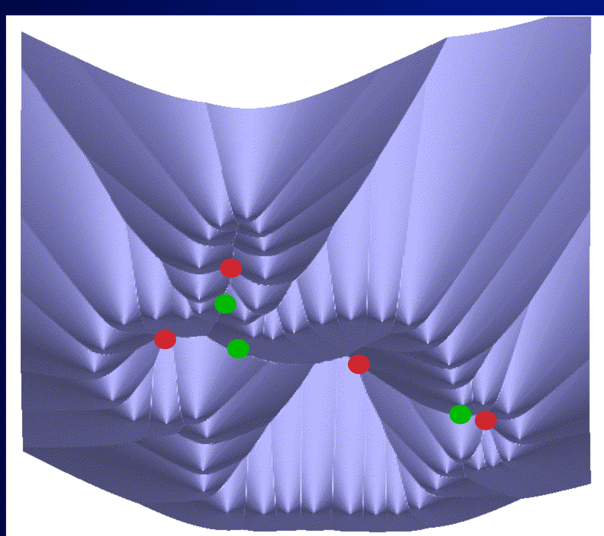
Distance from nearest sample.

Distance function



Consdier uphill flow …. Idea: interior is part that flows to interior maxima.

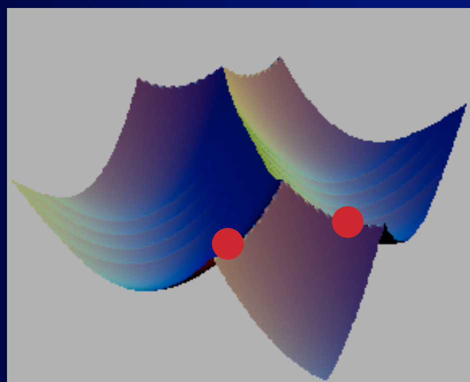
Distance function

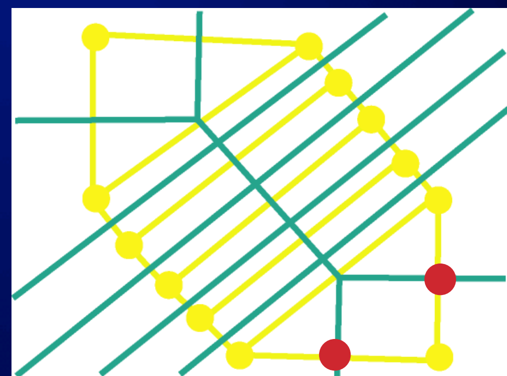


Compute flow combinatorially using Delaunay/ Voronoi

Max and (some) saddle points.

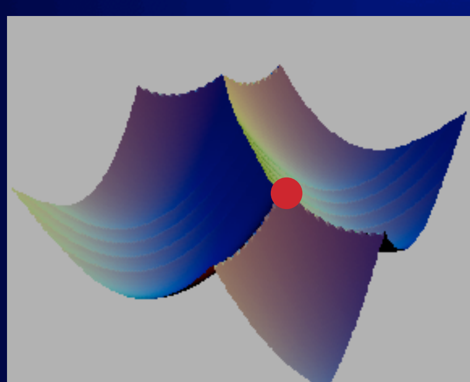
Distance function structure

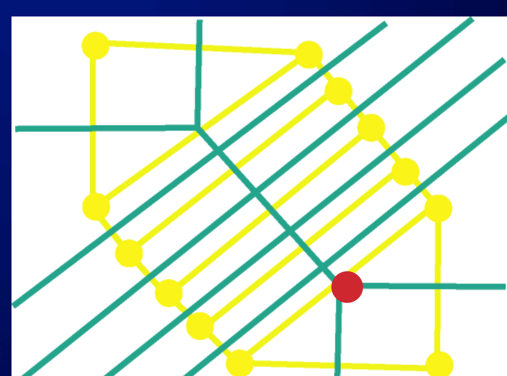




Critical points where dual Delaunay and Voronoi faces intersect.

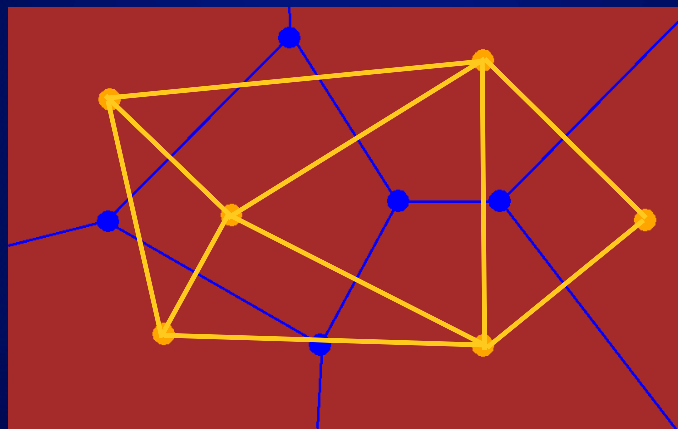
Distance function structure





Critical points where dual Delaunay and Voronoi faces intersect.

Not all pairs are critical



Wrap

Edelsbrunner - (95), Wrap, to appear....

Product!

Based on similar flow idea.

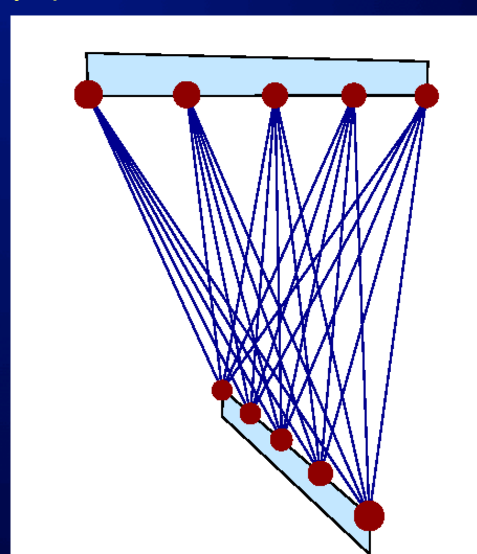
Running time

All $O(n^2)$ in theory because of complexity of 3D Delaunay triangulation. Practically, Delaunay is bottleneck.

But not in practice?

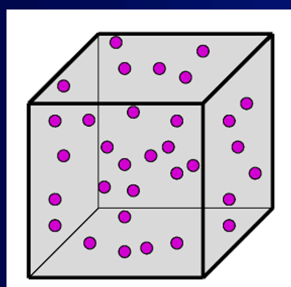
Delaunay complexity lower bound

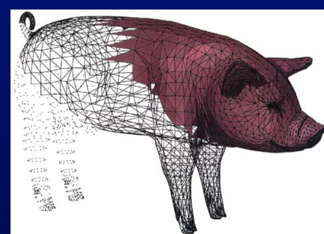
Arrange points on two skew line segments - $O(n^2)$ Delaunay triangulation

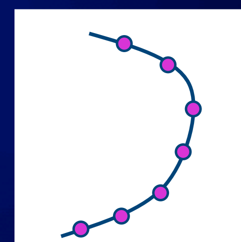


Dimension of distribution

Dwyer, 91: Random points in sphere have Delaunay triangulation of size O(n) in R^d.





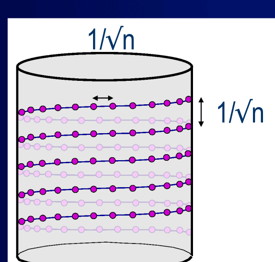


O(n) ?? $O(n^2)$

2D surface in R^3

Cylinder

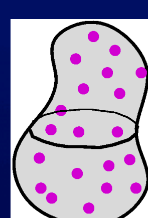
[Erickson01]



$O(n^{3/2})$

"Generic"

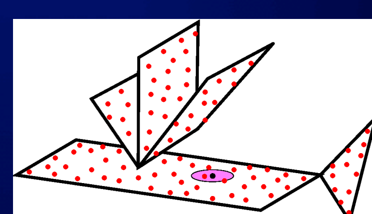
[Attali Boissonnat Lieutier 03]



O(n log n)

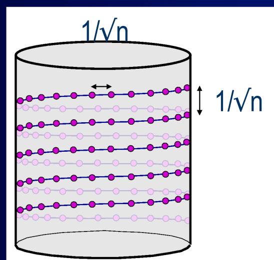
Polyhedron

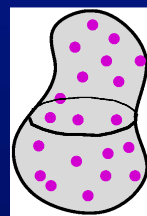
[Attali Boissonnat 02]

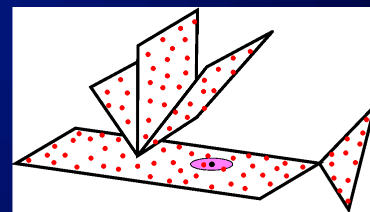


O(n)

Tangent spheres







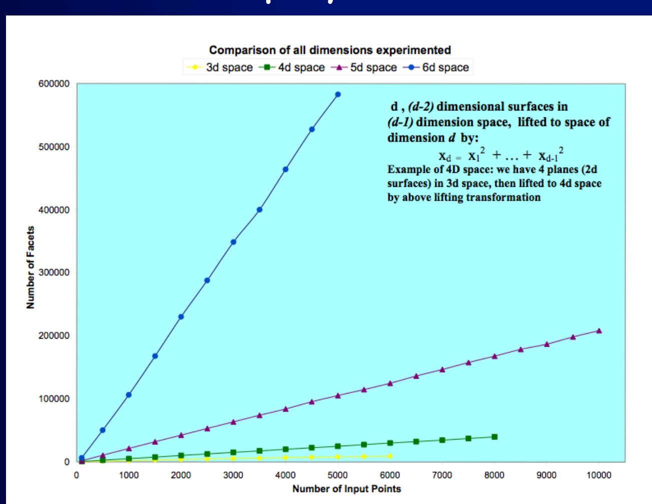
Infinite number of tangent spheres touch an infinite number of surface points.

All but a finite number of tangent spheres touch a finite number of surface points

All tangent spheres touch a finite number of surface points

Polyhedral (d-1) in R^d

Delauany triangulation of points on d-1 dimensional polyhedral surfaces:



simplices per sample:

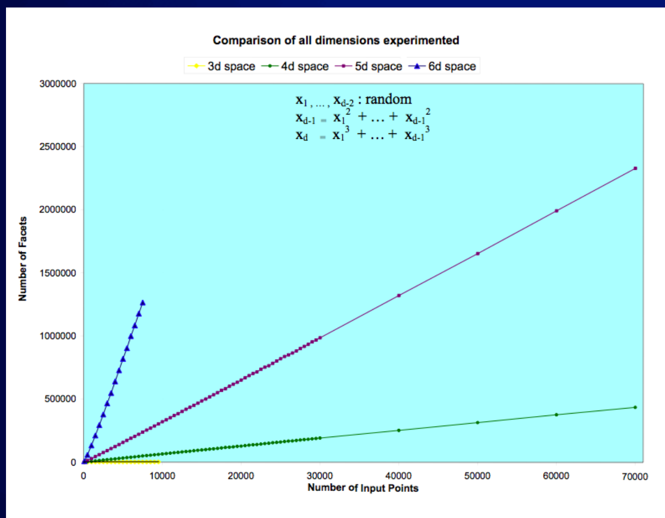
4D - 3

5D - 20

6D - 100

Generic (d-1) in R^d

Delauany of points on cubic polynomial



simplices per sample:

4D - 5

5D - 30

6D - 130

Upper bound, polyhedral

[A,Attali&Devillers] Number of Delaunay simplicies for n points nearly uniformly sampling all faces of a p-dimensional polyhedral surface (not nec. connected or convex) in R^d is:

$O(n^{(d+1-k)/p})$, $k = \text{ceiling}((d+1)/(p+1))$

Next week…

…maybe we'll get a chance to improve this.