Shape Analysis with the Delaunay Triangulation

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Shape of a Point Set
Surface Reconstruction

Input: Samples from object surface.
Output: Polygonal model.

Point Set Capture

Cyberware model 15
Point Grey Bumblebee
Applications

Delson et al, AMNH

Levoy et al, Stanford

Allen, Curless, Popovic, U Wash.

Power Diagram

Weighted Voronoi diagram. Input: balls.

\[ \text{Dist}(x, \text{ball}) = \text{dist}^2(x, \text{center}) - \text{radius}^2 \]

Polyhedral cells, same algorithm as regular Voronoi diagram (lift to convex hull)
Alpha-shapes

Overlay Voronoi diagram of balls on union of input balls. Discard features outside of the union.

Alpha-shapes

Alpha shape is the set of weighted Delaunay features dual to the weighted Voronoi features intersecting union of balls.
Alpha-shapes

Edelsbrunner, Kirkpatrick, Seidel, 83

Edelsbrunner, 93: Alpha shape is homotopy equivalent to union of balls, close correspondence with union structure.

Edelsbrunner & Muecke, 94: 3D surface reconstruction.

Alpha-shape reconstruction

Put small ball around each sample, compute alpha-shape.
Difficulty

Usually no ideal choice of radius.

Ball-pivoting

Bernardini et al, IBM

Fixed-radius ball “rolling” over points selects subset of alpha-shape.
**Medial Axis**

**Medial axis** is set of points with more than one closest surface point.

Blum, 67

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**3D Medial Axis**

Medial axis of a surface forms a dual surface.
Medial Axis

Maximal ball avoiding surface is a **medial ball**.

Every solid is a union of balls!

Relation to Voronoi

Voronoi balls approximate medial balls.

For dense surface samples in 2D, all Voronoi vertices lie near medial axis.

Ogniewicz, 92
Convergence

In 2D, set of Voronoi vertices converges to the medial axis as sampling density increases.

Discrete unions of balls

Voronoi balls approximate the object and its complement.
2D Curve Reconstruction

Blue Delaunay edges reconstruct the curve, **pink** triangulate interior/exterior.

Many algorithms, with proofs, for coloring edges.

2D Medial Reconstruction

**Pink** approximate medial axis.

By **nerve theorem**, approximation is homotopy equivalent to object and its complement.
3D Voronoi/Delaunay

Voronoi cells are convex polyhedra.

Voronoi balls pass through 4 samples.

Delaunay tetrahedra.

Sliver tetrahedra

In 3D, some Voronoi vertices are not near medial axis ...
Sliver tetrahedra

.... even when samples are arbitrarily dense.

Interior Voronoi balls

Poles

Subset of Voronoi vertices, the poles, approximate medial axis.

Interior polar balls

Amenta & Bern, 98 “Crust” papers
For dense surface samples, Voronoi cells are:

- long and skinny,
- perpendicular to surface,
- with ends near the medial axis.

Poles are Voronoi vertices at opposite ends.

To find: farthest Voronoi vertex from sample, farthest on opposite side.
Sampling Requirement

$\varepsilon$-sample: distance from any surface point to nearest sample is at most small constant $\varepsilon$ times distance to medial axis.

Note: surface has to be smooth.

Intuition: dense sampling where curvature is high or near features.
Large balls tangent

Any large ball (with respect to distance to medial axis) touching sample $s$ has to be nearly tangent to the surface at $s$.

Specifically

Given an $\varepsilon$-sample from a surface $F$:

Angle between normal to $F$ at sample $s$ and vector from $s$ to either pole = $O(\varepsilon)$
Results

Look for algorithms where....
Input: $\varepsilon$-sample from surface $G$
Output: PL-surface,

- near $G$, converges
- normals near $G$, converge
- PL manifold
- homeomorphic to $G$

Formal Algorithms

Amenta and Bern, crust
Amenta, Choi, Dey and Leekha, co-cone
Boissonnat and Cazals, natural neighbor
Amenta, Choi and Kolluri, power crust
**Co-cone**

Estimate normals, choose candidate triangles with good normals at each vertex.

Extract manifold from candidates.

**Co-cone**

Works well on clean data from a closed surface.

Amenta, Choi, Dey, Leekha 2000
Co-cone extensions

Dey & Giesen, undersampling errors.
Dey & Goswami, hole-filling.

Dey, Giesen & Hudson, divide and conquer for large data.

Power Crust

Amenta, Choi and Kolluri, 01

Idea: Approximate object as union of balls, compute polygonal surface from balls.
Power Crust

Start with all poles.

Compute polygonal decomposition using power diagram.
Power Crust

Label power diagram cells inside or outside object (skipping details).

Inside cells form polyhedral solid.

Power Crust

Boundary of solid gives output surface.

Connect inner poles with adjacent power diagram cells for approximate medial axis.
Example
Laser range data, power crust, simplified approximate medial axis.

Medial axis approximation
Dey & Zhao, 02
Voronoi diagram far from surface.
**Medial axis approximation**

Medial axis of union of balls = lower dimensional parts of alpha shape + intersection with Voronoi diagram of union vertices.

Attali & Montanvert, 97, A & Kolluri, 01

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**Distance function**

Giesen and John, 01,02

Distance from nearest sample.
Distance function

Consider uphill flow. Idea: interior is part that flows to interior maxima.

Distance function

Compute flow combinatorially using Delaunay/Voronoi

Max and (some) saddle points.
Critical points where dual Delaunay and Voronoi faces intersect.
Not all pairs are critical

Wrap

Edelsbrunner - (95), Wrap, to appear....

Product!
Based on similar flow idea.
Running time

All $O(n^2)$ in theory because of complexity of 3D Delaunay triangulation. Practically, Delaunay is bottleneck.

But not in practice?

Delaunay complexity lower bound

Arrange points on two skew line segments - $O(n^2)$ Delaunay triangulation
**Dimension of distribution**

Dwyer, 91: Random points in sphere have Delaunay triangulation of size $O(n)$ in $\mathbb{R}^d$.

- $O(n)$
- ??
- $O(n^2)$

**2D surface in $\mathbb{R}^3$**

- **Cylinder**
  - [Erickson01]
  - $O(n^{3/2})$

- **"Generic"**
  - [Attali Boissonnat Lieutier 03]
  - $O(n \log n)$

- **Polyhedron**
  - [Attali Boissonnat 02]
  - $O(n)$
Tangent spheres

Infinite number of tangent spheres touch an infinite number of surface points.

All but a finite number of tangent spheres touch a finite number of surface points.

All tangent spheres touch a finite number of surface points.

Polyhedral (d-1) in $\mathbb{R}^d$

Delauany triangulation of points on d-1 dimensional polyhedral surfaces:

# simplices per sample:

4D - 3
5D - 20
6D - 100
**Generic (d-1) in \( \mathbb{R}^d \)**

Delaunany of points on cubic polynomial

Number of simplices per sample:
- 4D - 5
- 5D - 30
- 6D - 130

Upper bound, polyhedral

[A, Attali & Devillers] Number of Delaunay simplices for \( n \) points nearly uniformly sampling all faces of a \( p \)-dimensional polyhedral surface (not nec. connected or convex) in \( \mathbb{R}^d \) is:

\[ O(n^{(d+1-k)/p}), \quad k = \text{ceiling}((d+1)/(p+1)) \]
Next week...

...maybe we'll get a chance to improve this.