II. Growth of Functions and Asymptotic Notations
Growth of Functions and Asymptotic Notations

Overview:

- Study a way to describe the growth of functions in the limit – *asymptotic efficiency*
- Focus on what’s important (leading factor) by abstracting lower-order terms and constant factors
- Indicate running times of algorithms
- A way to compare “sizes” of functions
  \[ O \approx \leq \]
  \[ \Omega \approx \geq \]
  \[ \Theta \approx = \]

In addition, \( o \approx < \) and \( \omega \approx > \)
\textit{O}-notation

- $g(n)$ is an asymptotic upper bound for $f(n)$:
  \[ f(n) = O(g(n)) \]
  if there exists constants $c$ and $n_0$ such that
  \[ 0 \leq f(n) \leq c \cdot g(n) \quad \text{for } n \geq n_0 \]

- Example:
  - $2n + 10 = O(n^2)$, by picking $c = 1$ and $n_0 = 5$
More on $O$-notation

- $O(g(n))$ is a set of functions

\[ O(g(n)) = \{ f(n) : \exists c, n_0 \text{ s.t. } 0 \leq f(n) \leq c \cdot g(n) \text{ for } n \geq n_0 \} \]

- Examples of functions in $O(n^2)$:
  - $n^2 + n$
  - $n^2 + 1000n$
  - $1000n^2 + 1000n$
  - $n/1000$
  - $n^2 / \lg n$
**Ω-notation**

- \( g(n) \) is an asymptotic lower bound for \( f(n) \).

\[
f(n) = \Omega(g(n))
\]

if there exists constants \( c \) and \( n_0 \) such that

\[
0 \leq c \cdot g(n) \leq f(n) \quad \text{for } n \geq n_0
\]

- Example:
  
  \[
  \sqrt{n} = \Omega(\lg n) \quad \text{by picking } c = 1 \text{ and } n_0 = 16
  \]
More on $\Omega$-notation

- $\Omega(g(n))$ is a set of functions

  $$\Omega(g(n)) = \{ f(n) : \exists c, n_0 \ \text{s.t.} \ 0 \leq c \cdot g(n) \leq f(n) \ \text{for} \ n \geq n_0 \}$$

- Examples of functions in $\Omega(n^2)$:
  - $n^2$
  - $n^2 + n$
  - $n^2 - n$
  - $1000n^2 + 1000n$
  - $1000n^2 - 1000n$
  - $n^{2.00001}$
  - $n^2 \lg n$
  - $n^3$
**Θ-notation**

- $g(n)$ is an asymptotic tight bound for $f(n)$.

  $$f(n) = \Theta(g(n))$$

  if there exists constants $c_1$, $c_2$ and $n_0$ such that

  $$0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 g(n)$$

  for $n \geq n_0$.

- Example:
  - $\frac{1}{2}n^2 - 2n = \Theta(n^2)$, pick $c_1 = \frac{1}{4}$
    $c_2 = \frac{1}{2}$ and $n_0 = 8$.
  - If $p(n) = \sum_{i=1}^{d} a_i n^i$ and $a_d > 0$, then $p(n) = \Theta(n^d)$
More on $\Theta$-notation

- $\Theta(g(n))$ is a set of functions

\[
\Omega(g(n)) = \\
\{ f(n) : \exists c_1, c_2, n_0 \text{ s.t. } 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 g(n) \text{ for } n \geq n_0 \}
\]

- Examples of functions in $\Theta(n^2)$:
  - $n^2$
  - $n^2 + n$
  - $n^2 - n$
  - $1000n^2 + 1000n$
  - $1000n^2 - 1000n$
Theorem.

$O$ and $\Omega$ iff $\Theta$. 
Using limits for comparing orders of growth

In order to determine the relationship between \( f(n) \) and \( g(n) \), it is often useful to examine

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = L
\]

The possible outcomes:

1. \( L = 0 \): \( f(n) = O(g(n)) \)
2. \( L = \infty \): \( f(n) = \Omega(g(n)) \)
3. \( L \neq 0 \) is finite: \( f(n) = \Theta(g(n)) \)
4. There is no limit: this technique cannot be used to determine the asymptotic relationship between \( f(n) \) and \( g(n) \).
L’Hopital’s rule. Let \( f(x) \) and \( g(x) \) be differential functions with derivatives \( f'(x) \) and \( g'(x) \), respectively, such that

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty.
\]

Then

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}.
\]
Examples

1. $f(n) = n^2$ and $g(n) = n \lg n$

   $n^2 = \Omega(n \lg n)$

2. $f(n) = n^{100}$ and $g(n) = 2^n$

   $n^{100} = O(2^n)$

3. $f(n) = 10n(n + 1)$ and $g(n) = n^2$

   $10n(n + 1) = \Theta(n^2)$