Finding the closest pair of points in 1-dimension

Problem statement:\(^1\)

*Given a set \( S \) of \( n \) points on a line (unsorted), find two points whose distance is smallest.*

\(^1\)Section 33.4 of [CLRS, 3rd edition] provides an algorithm for finding the closest pair of points in 2-dimension, i.e., on a plane.
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A brute-force solution

- Pick two of \( n \) points and compute the distance
- Cost:

\[
T(n) = \binom{n}{2} = \frac{n!}{2!(n-2)!} = \Theta(n^2).
\]
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Algorithm 1

1. Sort the points, say Merge Sort
2. Perform a linear scan
3. Cost: $\Theta(n \lg n) + \Theta(n) = \Theta(n \lg n)$

Remark: Unfortunately, Algorithm 1 cannot be extended to the 2-dimension case.
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Algorithm 2 (Divide-and-Conquer):

1. Divide the set $S$ by some point $mid$ (say, median) into two sets $S_1$ and $S_2$, with the property:

   $$p < q \text{ for all } p \in S_1 \text{ and } q \in S_2$$

2. Conquer: (a) finds the closest pair recursively on $S_1$ and $S_2$, gives us two pairs of points $\{p_1, p_2\}$ and $\{q_1, q_2\}$, the closest pair in $S_1$ and $S_2$, respectively.
   (b) finds the closest crossing pair $\{p_3, q_3\}$ with $p_3 \in S_1$ and $q_3 \in S_2$.

3. Combine: the closest pair in the set $S$ is

   $$\text{argmin}\{|p_1 - p_2|, |q_1 - q_2|, |p_3 - q_3|\}.$$
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Remark 1:

1. Both $p_3$ and $q_3$ must be within distance $d = \min\{|p_1 - p_2|, |q_1 - q_2|\}$ of $\text{mid}$ if $\{p_3, q_3\}$ is to have a distance smaller than $d$.

2. How many points of $S_1$ can lie in $(\text{mid} - d, \text{mid}]$?
   
   \textit{answer: at most one}

3. How many points of $S_2$ can lie in $[\text{mid}, \text{mid} + d)$?
   
   \textit{answer: at most one}

4. Therefore, the number of pairwise comparisons that must be made between points in different subsets is thus \textit{at most one}. 
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ClosestPair(S)
if |S| = 2, then
else
    if |S| = 1
        d = infty
    else
        m = median(S)
        construct S1 and S2
        d1 = ClosestPair(S1)
        d2 = ClosestPair(S2)
        p = max(S1)
        q = min(S2)
        d = min(d1, d2, q-p)
    end if
end if
return d
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Remark 2

1. A median is the “halfway point” of the set $A$ can be found in linear time $\Theta(n)$ on average (see Chapter 9).
2. The points in the intervals $(\text{mid} - d, \text{mid}]$ and $[\text{mid}, \text{mid} + d)$ can be found in linear time $O(n)$.
3. Complexity for finding the closest pair of points in 1-dimension:

$$T(n) = 2 T\left(\frac{n}{2}\right) + \Theta(n) = \Theta(n \lg n).$$