Applications of DFS

1. For a undirected graph,
   (a) a DFS produces only Tree and Back edges
   (b) acyclic (tree) iff a DFS yeilds no back edges

2. A directed graph is acyclic iff a DFS yields no back edges, i.e.,
   \[ \text{dag} \Leftrightarrow \text{no back edges} \]

3. Topological sort of a dag (= directed acyclic graph) – next

4. Connected components of a undirected graph (see Homework 6)

5. Strongly connected components of a directed graph (see Sec.22.5 of [CLRS,3rd ed.])
Topological sort

- A topological sort (TS) of a dag $G = (V, E)$ is a linear ordering of all its vertices such that if $(u, v) \in E$, then $u$ appears before $v$.
- A TS is not possible if $G$ has a cycle.
- The ordering is not necessarily unique.
Topological sort

Application: call-graph
Topological sort

- TS algorithm
  1. run DFS(G) to compute finishing times $f[v]$ for all $v \in V$
  2. output vertices in order of decreasing times

- Running time: $\Theta(|V| + |E|)$
Topological sort

Example: Getting-dressed-graph and DFS

Topologically sorted

Topologically sorted

The following simple algorithm topologically sorts a dag:

TOPOLOGICAL-SORT(G)
1. call DFS(G) to compute finishing times
2. for each vertex v in order of decreasing finishing time, insert v onto the front of a linked list
3. return the linked list of vertices

We can perform a topological sort in time O(V + E), since depth-first search
takes O(V + E) time and it takes O(1) time to insert each of the
|V| vertices onto the front of the linked list.

We prove the correctness of this algorithm using the following key lemma characterizing directed acyclic graphs.
Topological sort

**Theorem (correctness of the algorithm):**
TS(G) produces a topological sort of a dag G.

Proof: *Just need to show that if* \((u, v) \in E\), *then* \(f[v] < f[u]\).

When we explore edge \((u, v)\), u is gray, what’s the color of v?

- Is v gray too?
  - no, because then v would be ancestor of u, edge \((u, v)\) is a back edge, a contradiction of a dag.

- Is v white?
  - yes, then v is descendant of u, by DFS, \(d[u] < d[v] < f[v] < f[u]\)

- Is v black?
  - yes, then v is already finished. Since we’re exploring \((u, v)\), we have not yet finished u, therefore \(f[v] < f[u]\)