4. How to prove a problem is NPC

Recall

- A decision problem $A$ is **NP-complete (NPC)** if

  1. $A \in \text{NP}$ and
  2. every other problems $B$ in NP is *polynomially reducible* to $A$, denoted as

    $$B \leq_T A$$
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Recall

Polynomial reduction $B \leq_T A$

Let $A$ and $B$ be two decision problems, $B$ is polynomially reducible to $A$, if there is a poly-time computable transformation $T$ such that

Yes-instance of $A \iff$ Yes-instance of $B$
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- The reducibility relation “≤ₜ” is transitive, i.e,

\[ A \leqₜ B \quad \text{and} \quad B \leqₜ C \quad \text{imply} \quad A \leqₜ C \]

- Therefore, to prove that a problem \( A \) is NPC:
  1. show that \( A \in \text{NP} \)
  2. choose some known NPC problem \( B \)
     - define a polynomial transformation \( T \) from \( B \) to \( A \)
     - show that \( B \leqₜ A \)

- Why sufficient? the logic is as follows:

Since \( B \) is NPC, all problems in NP is reducible to \( B \).
Show \( B \) is reducible to \( A \).
Then all problems in NP is reducible to \( A \).
Therefore, \( A \) is NPC
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Example 1:
Prove that Undirected HC is NPC.

Proof:
- (1) undirected HC is in NP
- (2)
  - The directed HC is known to be NPC (taken as a fact)
  - Next we show that

\[
\text{directed HC} \leq_T \text{undirected HC}
\]

- By (1) and (2), we conclude that the undirected HC is NPC.
4. How to prove a problem is NP-complete

Example 1, cont’d:
Show that

\[
\text{directed } \text{HC} \leq_T \text{undirected } \text{HC}
\]

- Define transformation:

  Let \( G = (V, E) \) be a directed graph. Define \( G' \) to the undirected graph \( G' = (V', E') \) by the following transformation \( T \):
  
  - \( v \in V \rightarrow v^1, v^2, v^3 \in V' \) and \( (v^1, v^2), (v^2, v^3) \in E' \)
  
  - \( (u, v) \in E \rightarrow (u^3, v^1) \in E' \)

- \( T \) is polynomial-time computable.

- Now, we just need to show that

  \( G \) has a HC \( \iff \) \( G' \) has a HC.
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Example 1, cont’d

“⇒” Suppose that \( G \) has a directed HC: \( v_1, v_2, \ldots, v_n, v_1 \)
Then
\[
v_{1}^{1}, v_{1}^{2}, v_{1}^{3}, v_{2}^{1}, v_{2}^{2}, v_{2}^{3}, \ldots, v_{n}^{1}, v_{n}^{2}, v_{n}^{3}, v_{1}^{1}
\]
is an undirected HC for \( G' \).

“⇐” 1. Suppose that \( G' \) has an undirected HC, the three vertices \( v^1, v^2, v^3 \) that correspond to one vertex from \( G \) must be traversed \textbf{consecutively} in the order \( v^1, v^2, v^3 \) or \( v^3, v^2, v^1 \), since \( v^2 \) cannot be reached from any other vertex in \( G' \).
2. Since the other edges in \( G' \) connect vertices with superscripts 1 or 3, if for any one triple the order of the superscripts is 1, 2, 3, then the order is 1, 2, 3 for all triples. Otherwise, it is 3, 2, 1 for all triples.
3. Therefore, we may assume that the undirected HC of \( G' \) is
\[
\underbrace{v_{i_{1}}^{1}, v_{i_{1}}^{2}, v_{i_{1}}^{3}}_{i_{1}}, \underbrace{v_{i_{2}}^{1}, v_{i_{2}}^{2}, v_{i_{2}}^{3}}_{i_{2}}, \ldots, \underbrace{v_{i_{n}}^{1}, v_{i_{n}}^{2}, v_{i_{n}}^{3}}_{i_{n}}, v_{i_{1}}^{1}.
\]
Then
\[
v_{i_{1}}, v_{i_{2}}, \ldots, v_{i_{n}}, v_{i_{1}}\text{ is a directed HC for } G.
\]
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Example 2: Show that

\[
\text{Subset-Sum} \leq_T \text{Set-Partition}
\]

Since Subset-Sum is known to be NPC, the above reduction implies that Set-Partition is also NPC.

Subset-Sum decision problem:

Given a positive integer \( c \), and a set \( S = \{s_1, s_2, \ldots, s_n\} \) of positive integers \( s_i \) for \( i = 1, 2, \ldots, n \). Is there a \( J \subseteq \{1, 2, \ldots, n\} \) such that \( \sum_{i \in J} s_i = c \)? Assume that \( w = \sum_{i=1}^{n} s_i \geq c \).

Set-Partition decision problem:

Given a set \( S \) of numbers. Can \( S \) be partitioned into two sets \( A \) and \( \bar{A} = S - A \) such that \( \sum_{x \in A} x = \sum_{x \in \bar{A}} x \)?
4. How to **prove** a problem is NPC

Example 2, cont’d

- Let \( S \) be an instance of Subset-Sum with \( w = \sum_{s_i \in S} s_i \) and the target \( c \).

- Define the set \( S' \) (i.e., the transformation \( T \) from \( S \) to \( S' \)) as follows:

  \[
  S' = S \cup \{u, v\}, \quad \text{where} \quad u = 2w - c, \quad v = w + c.
  \]

- Next to show that

  Yes of Subset-Sum of \( S \) \( \iff \) Yes of Set-Partition of \( S' \)
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Example 2, cont’d

⇒ Let $J \subseteq S$ and the elements in $J$ sum to $c$. Then $J \cup \{u\}$ sum to $2w$. Note that the elements in $\overline{J} = S - J$ sum to $w - c$. Hence, $\overline{J} \cup \{v\}$ also sums to $2w$. Therefore, $S'$ can be partitioned into $J \cup \{u\}$ and $\overline{J} \cup \{v\}$ where both partitions sum to $2w$. Thus, Yes of Subset-Sum transforms to a Yes of Set-Partition.
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Example 2, cont’d

⇐ Assume $S'$ can be partitioned into two sets, $T$ and $\overline{T} = S' - T$, such that

\[
\sum_{x \in T} x = \sum_{x \in \overline{T}} x. \tag{1}
\]

Since $w + u + v = 4w$, the sum of the elements in both sets must be equal to $2w$. Therefore, $u$ must be in one set and $v$ must be in the other because $u + v = 3w$. Without loss of generality, let $u \in T$. Then

\[
2w = \sum_{x \in T} x = u + \sum_{x \in T - u} x = 2w - c + \sum_{x \in T - u} x.
\]

It implies that

\[
\sum_{x \in T - u} x = c
\]

Thus, Yes of Set-Partition *transforms* to Yes of Subset-Sum.
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Additional examples/exercises:

- **Example 3:**
  Graph 3-COLOR $\leq_T$ 4-COLOR (*homework 8*)

- **Example 4:**
  Subset Sum $\leq_T$ Job Scheduling (*handout*)