VII. Graph Algorithms

Notion of graphs

Basic terminology

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- Reading: Appendix B.4, pp.1168-1172 of [CLRS,3rd ed.]

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Representing a graph by an Adjacency Matrix A

- $A = (a_{ij})$ is a $|V| \times |V|$ matrix, where

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- A is typically very sparse
  use a sparse storage scheme in practice

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Representing a graph by an Incidence Matrix B

- $B = (b_{ij})$ is a $|V| \times |E|$ matrix, where

$$b_{ij} = \begin{cases} 1, & \text{if edge } e_j \text{ enters vertex } v_i \\ -1, & \text{if edge } e_j \text{ leaves vertex } v_i \\ 0, & \text{otherwise} \end{cases}$$

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  Answer: $\Theta(|V| + |E|)$ ("sparse representation")

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