# Dynamic Programming

Four-step (two-phase) method:

- 1. Characterize the structure of an optimal solution
- 2. Recursively define the value of an optimal solution
- 3. Compute the value of an optimal solution in a bottom-up fashion
- 4. Construct an optimal solution from computed information

# Longest Common Subsequence (LCS) – DP case study 3

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Output: longest common subsequence (LCS) of  $X_m$  and  $Y_n$ 

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  - ► Given  $X_7 = \langle A, B, C, B, D, A, B \rangle$  $Y_6 = \langle B, D, C, A, B, A \rangle$
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- 4. Longest common subsequence (LCS), e.g.
  - $ightharpoonup Z_4$  is a longest common subsequence (LCS) of  $X_7$  and  $Y_6$
  - ▶ LCS is not unique,  $\langle B, C, A, B \rangle$  is also a LCS.

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- ▶ Running time:  $\Theta(n \cdot 2^m)$
- ► Intractable!

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we have

$$c[i, j] = \max\{c[i, j - 1], c[i - 1, j]\}$$

In summary,

$$c[i,j] = \left\{ \begin{array}{ll} 0 & \text{if } i = 0 \text{ or } j = 0 \text{ (initials)} \\ c[i-1,j-1] + 1 & \text{if } x[i] = y[j] & \text{(Case 1)} \\ \max\{c[i,j-1],c[i-1,j]\} & \text{if } x[i] \neq y[j] & \text{(Case 2)} \end{array} \right.$$

In summary,

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 $\blacktriangleright$  Meanwhile, create b[i,j] to record the optimal subproblem solution chosen when computing c[i,j]

DP – step 3: compute c[i, j] (and b[i, j]) in a bottom-up approach

- ▶ Compute c[i,j] and b[i,j] in a bottom-up approach.
  - c[i, j] is the length of LCS $(X_i, Y_j)$
  - $lackbox{b}[i,j]$  shows how to construct the corresponding  $\mathsf{LCS}(X_i,Y_j)$

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#### Cost:

- Running time:  $\Theta(mn)$
- ▶ Space:  $\Theta(mn)$

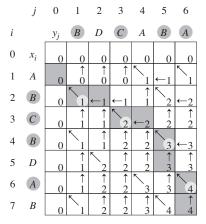
```
LCS-length(X,Y)
set c[i,0] = 0 and c[0,j] = 0
for i = 1 to m // Row-major order to compute c and b arrays
   for j = 1 to n
       if X(i) = Y(j)
          c[i,j] = c[i-1,j-1] + 1
          b[i,i] = 'Diag' // go to up diagonal
       elseif c[i-1,j] >= c[i,j-1]
          c[i,i] = c[i-1,i]
          b[i,j] = 'Up' // go up
       else
          c[i,j] = c[i,j-1]
          b[i,j] = 'Left' // go left
       endif
   endfor
endfor
return c and b
```

DP - step 4: construct an optimal solution from computed information

Example:  $X_7 = \langle A, B, C, B, D, A, B \rangle$  and  $Y_6 = \langle B, D, C, A, B, A \rangle$ 

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$$c[\cdot,\cdot] + b[\cdot,\cdot] : \qquad \qquad j \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$i \quad y_j \quad B \quad D \quad C \quad A \quad B \quad A$$

$$0 \quad x_i \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

$$1 \quad A \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad \leftarrow 1 \quad 1$$

$$2 \quad B \quad 0 \quad 1 \quad \leftarrow 1 \quad 1 \quad 2 \quad \leftarrow 2$$

$$3 \quad C \quad 0 \quad 1 \quad 1 \quad 2 \quad \leftarrow 2 \quad 2 \quad 2$$

$$4 \quad B \quad 0 \quad 1 \quad 1 \quad 2 \quad \leftarrow 2 \quad 2 \quad 2$$

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$$5 \quad D \quad 0 \quad 1 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \quad \leftarrow 3$$

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$$6 \quad A \quad 0 \quad 1 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \quad \leftarrow 3$$

$$7 \quad B \quad 0 \quad 1 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \quad \leftarrow 4$$

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- (1) Length of LCS = c[7,6] = 4
- (2) By the b-table (" $\uparrow$ ,  $\leftarrow$ ,  $\nwarrow$ "), the LCS is BCBA

