VIII. NP-completeness

NP-Completeness – overview

1. Introduction
2. P and NP
3. NP-complete (NPC): formal definition
4. How to prove a problem is NPC
5. How to solve a NPC problem: approximate algorithms

1. Introduction

Tractable and intractable problems

1. Introduction

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1. Introduction

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Almost all the algorithms we have studied thus far have been polynomial-time algorithms on inputs of size n, their worst-case running time is $O(n^k)$ for some constant k.

1. Introduction

NP-complete (NPC) problems: an informal definition

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A large class of very diverse problems share the following properties:

1. We *only know* how to solve those problems in time much larger than polynomial, namely exponential time.

2. If we could *solve one NPC porblem* in polynomial time, then there is a way to *solve every NPC problem* in polynomial time.

1. Introduction

Reasons to study NPC porblems – practical

1. Introduction

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1. Introduction

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1. Introduction

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1. Introduction

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- you can use a known algorithm for an intractable problem, and accept that it will take a long long time to solve; or
- you can settle for approximating the solution, e.g., finding a nearly best solution rather than the optimum; or
- you can change your problem formulation so that it is solvable in polynomial time.

1. Introduction

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Whether NPC problems have polynomial solutions?

- First posed in 1971
http://www.claymath.org/millennium-problems

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P-vs-NP Examples

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1. Introduction

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1. Introduction

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- Shortest path:
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  finding the longest *simple* path between two vertices in a directed graph.

The first one is solvable in polynomial time (the Bellman-Ford algorithm), and the second is NPC, but the difference appears to be slight.

1. Introduction

P-vs-NP Examples

[1] Euler cycle of $G = (V, E)$ **iff** in-degree(v) = out-degree(v) for $\forall v \in V$.

1. Introduction

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Example 2.

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1. Introduction

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1. Introduction

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1. Introduction

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Example 3.

- Minimum spanning tree (MST):
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1. Introduction

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1. Introduction

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- Minimum spanning tree (MST):
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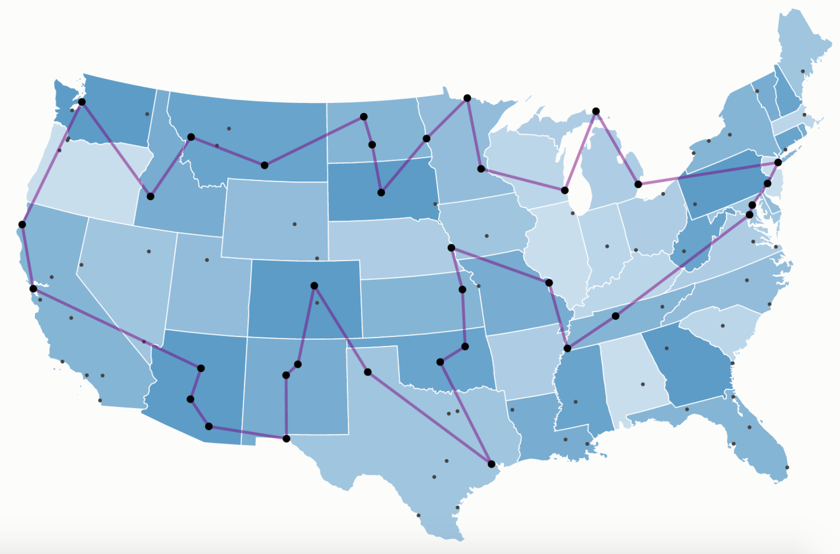
1. Introduction

Traveling salesperson problem (TSP)



1. Introduction

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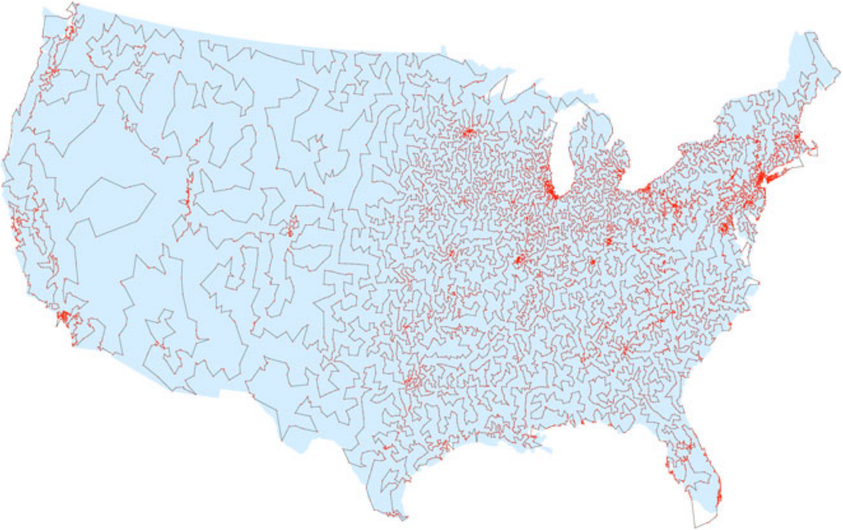

1. Introduction

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1. Introduction

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Example 4.

- Circuit value:
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1. Introduction

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1. Introduction

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- Circuit value:
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1. Introduction

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Remarks:

- *To simplify discussion, we can consider only decision problems, rather than optimization problems.*
- The optimization problems are at least as hard to solve as the related decision problems, we have not lost anything essential by doing so.

1. Introduction

Optimization-vs-Decision Examples

1. Introduction

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Example 1

Graph coloring: A coloring of a graph $G = (V, E)$ is a mapping

$$C : V \to S$$

where S is a finite set of "colors", such that

$$(u, v) \in E \Rightarrow C(u) \neq C(v)$$

1. Introduction

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1. Introduction

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$$(u, v) \in E \Rightarrow C(u) \neq C(v)$$

- optimization problem: given G, determine the smallest number of colors needed.
- decision problem: given G and a positive integer k, is there a coloring of G using at most k colors?

1. Introduction

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1. Introduction

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Example 2.

> *Hamiltonian cycle: A Hamiltonian cycle is cycle that passes through every vertex exactly once.*

1. Introduction

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1. Introduction

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> *Hamiltonian cycle: A Hamiltonian cycle is cycle that passes through every vertex exactly once.*

- decision problem: Does a given graph have a Hamiltonian cycle?
- optimization problem: Give a list of vertices of a Hamiltonian cycle.

1. Introduction

Optimization-vs-Decision Examples

1. Introduction

Optimization-vs-Decision Examples

Example 3.

> *TSP (Traveling Salesperson Problem): given a weighted graph and an integer k, is there a cycle that visits all vertices exactly once (Hamiltonian cycle) whose total weight is k or less?*

1. Introduction

Optimization-vs-Decision Examples

Example 3.

> *TSP (Traveling Salesperson Problem): given a weighted graph and an integer k, is there a cycle that visits all vertices exactly once (Hamiltonian cycle) whose total weight is k or less?*

- optimization problem: given a weighted graph, find a minimum Hamiltonian cycle.

1. Introduction

Optimization-vs-Decision Examples

Example 3.

> *TSP (Traveling Salesperson Problem): given a weighted graph and an integer k, is there a cycle that visits all vertices exactly once (Hamiltonian cycle) whose total weight is k or less?*

- optimization problem: given a weighted graph, find a minimum Hamiltonian cycle.
- decision problem: given a weighted graph and an integer k, is there a Hamiltonian cycle with total weight at most k?

1. Introduction – recap

1. Tractable and intractable problems
   polynomial-boundness: $O(n^k)$
2. NP-complete problems – informal definition
3. P vs NP
   difference may appear "*only slightly*"
4. Optimization problems and decision problems