Matrix-matrix multiplication

- Problem:

  Given $n \times n$ matrices A and B, compute the product $C = A \cdot B$.

- Traditional method: (i, j, k)-triple-loop

```
for i = 1 to n
    for j = 1 to n
        C(i,j) = 0
        for k = 1:n
            C(i,j) = C(i,j) + A(i,k)*B(k,j)
        end
    end
end
```

- Complexity:

$$T(n) = \sum_{i=1}^{n}(\sum_{j=1}^{n}(\sum_{k=1}^{n} 2)) = 2n^3 = \Theta(n^3)$$

Matrix-matrix multiplication

- Divide-and-conquer: a naive implementation
  1. partition and then direct block multiplication

$$
\begin{aligned}
C &= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \\
&= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\
&= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}
\end{aligned}
$$

  2. Complexity:

$$T(n) = 8 \cdot T(\frac{n}{2}) + \Theta(n^2) = \Theta(n^3).$$

Same cost as the traditional method, No improvement

Matrix-matrix multiplication

- Strassen's divide-and-conquer method (Strassen's method): reduces the complexity to

$$T(n) = \Theta(n^{\lg 7}) \approx \Theta(n^{2.8074}).$$

- Reference:
  V. Strassen, *Gaussian elimination is not optimal*. Numer. Math. Vol.13, pp.354-356, 1969
- The subsequent improvements, with the current world record being $O(n^{2.37})$, are much more complicated (and astonishing), but less practical.

Matrix-matrix multiplication

- Strassen's method – Step 1: Divide

$$A = \begin{array}{c} \frac{n}{2} \\ \frac{n}{2} \end{array} \overset{\begin{array}{cc} \frac{n}{2} & \frac{n}{2} \end{array}}{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}} \quad \text{and} \quad B = \begin{array}{c} \frac{n}{2} \\ \frac{n}{2} \end{array} \overset{\begin{array}{cc} \frac{n}{2} & \frac{n}{2} \end{array}}{\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}}$$

Matrix-matrix multiplication

- Strassen's method – Step 2: Compute 10 matrices by $\pm$ only:

$$
\begin{aligned}
S_1 &= B_{12} - B_{22}\\
S_2 &= A_{11} + A_{12}\\
S_3 &= A_{21} + A_{22}\\
S_4 &= B_{21} - B_{11}\\
S_5 &= A_{11} + A_{22}\\
S_6 &= B_{11} + B_{22}\\
S_7 &= A_{12} - A_{22}\\
S_8 &= B_{21} + B_{22}\\
S_9 &= A_{11} - A_{21}\\
S_{10} &= B_{11} + B_{12}
\end{aligned}
$$

Matrix-matrix multiplication

- Strassen's method – Step 3: Compute 7 matrices by multiplication:

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 \\
P_2 &= S_2 \cdot B_{22} \\
P_3 &= S_3 \cdot B_{11} \\
P_4 &= A_{22} \cdot S_4 \\
P_5 &= S_5 \cdot S_6 \\
P_6 &= S_7 \cdot S_8 \\
P_7 &= S_9 \cdot S_{10}
\end{aligned}
$$

Matrix-matrix multiplication

- Strassen's method – Step 4: Add and subtract the P_i to construct submatrices C_{ij} of the product $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$:

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

Matrix-matrix multiplication

- Correctness: straightfoward verification
- Strassen's method – complexity

$$T(n) = 7 \cdot T(\frac{n}{2}) + \Theta(n^2) = \Theta(n^{\lg 7}).$$