1. (a) Prove that \((n + 3)^3 = \Theta(n^3)\)
(b) Is \(2^{n+1} = O(2^n)\)? why?
(c) Is \(2^{2n} = O(2^n)\)? why?

2. Order the following functions into a list such that if \(f(n)\) comes before \(g(n)\) in the list then \(f(n) = O(g(n))\). If any two (or more) of the same asymptotic order, indicate which.
   (a) Start with these basic functions
   \(n, 2^n, n \log n, n^3, n - n^3 + 7n^5, n^2 + \log n\)
   (b) Combine the following functions into your answer for part (a). Assume that \(0 < \epsilon < 1\).
   \(e^n, \sqrt{n}, 2^{n-1}, \log \log n, (\sqrt{2})^{\log n}, \log n, (\log n)^2, n!, n^{1+\epsilon}, 1\)

3. Recall that we have discussed the method of explicit substitution to solve a recurrence relation. It expands out the recurrence a few times until a pattern emerges. For instance, let us start with the recurrence
   \[T(n) = 2T(n/2) + cn.\]
   where \(c\) is a positive constant. By repeatedly applying this rule, we can bound \(T(n)\) in terms of \(T(n/2)\), then \(T(n/2^2)\), and so on, at each step getting closer to the basis value of \(T(1) = O(1)\):
   
   \[
   T(n) = 2T(n/2) + cn = 2[2T(n/2^2) + cn/2] + cn = 2^2T(n/2^2) + 2cn = 2^2[2T(n/2^3) + cn/2^2] + 2cn = 2^3T(n/2^3) + 3cn = \ldots
   \]
   A pattern is emerging. The general term is \(T(n) = 2^kT(n/2^k) + kcn.\) Plugging in \(k = \log n\), we get \(T(n) = n \cdot T(1) + cn \log n = \Theta(n \log n)\).

   Do the same thing for the following recurrence
   \[T(n) = 3 \cdot T\left(\frac{n}{2}\right) + cn.\]
   (a) What is the general \(k\)th term in this case?
   (b) What value of \(k\) should be plugged in to get the answer?

4. Give asymptotic upper and lower bounds for \(T(n)\) in each of the following recurrences. Assume that \(T(n)\) is constant for sufficient small \(n\), and \(c\) is a constant. Make your bounds as tight as possible, and justify your answers.
   (a) \(T(n) = T(n-1) + 1/n\)
   (b) \(T(n) = T(n-1) + c^n\), where \(c > 1\) is some constant
   (c) \(T(n) = 2T(n-1) + 1\)
   (d) \(T(n) = 2T\left(\frac{n}{2}\right) + \sqrt{n}\)
   (e) \(T(n) = 27T\left(\frac{n}{3}\right) + cn^3\)
   (f) \(T(n) = 5T\left(\frac{n}{4}\right) + cn^2\)

5. Use the master theorem to give tight asymptotic bounds for the following recurrences.
   (a) \(T(n) = 2T(n/4) + 1\)  (b) \(T(n) = 2T(n/4) + \sqrt{n}\)
   (c) \(T(n) = 2T(n/4) + n\)  (d) \(T(n) = 2T(n/4) + n^2\)