1. The money changing problem starts with a given set of positive integers called denominations \( x_1, x_2, \ldots, x_n \) (think of them as the integers 1, 5, 10, and 25) and an integer \( A \), we want to find nonnegative integers \( a_1, \ldots, a_n \geq 0 \) such that

\[
A = \sum_{i=1}^{n} a_i x_i.
\]

2. First, we note that \( A \) can be expressed as a linear combination of the \( x_i \) if and only if \( x_i = 1 \) for some \( i \). Here is a proof.

If one of your denominations \( x_i \) is 1, you will certainly be able to express every integer \( A \) as \( \sum_{i=1}^{n} a_i x_i \) for some nonnegative integers \( a_1, \ldots, a_n \). Conversely, in order to express \( A = 1 \) as a linear combination, you must have \( x_i = 1 \) for some \( i \).

3. In general (not necessarily satisfying the condition in the first part), a necessary condition that \( A = \sum_{i=1}^{n} a_i x_i \) is that \( g = \gcd(x_1, \ldots, x_n) \) divides \( A \). In fact, \( g|A \) turns out to be both necessary and sufficient for \( A \geq X \) for some (large) \( X \). Here is a proof.

From the extended Euclidean algorithm we know we can write \( g = \sum_{i=1}^{n} g_i x_i \) with some possibly negative \( g_i \). Now let \( G = \sum_{i=1}^{n} |g_i| x_i, x_{\min} = \min_i x_i, k = x_{\min}/g, \) and \( X = kG \).

First note that the \( k \) consecutive multiples of \( g \) in the set \( S = \{kG, kG + g, kG + 2g, \ldots, kG + (k-1)g\} \) all have nonnegative coefficients when written as \( \sum_{i=1}^{n} a_i x_i \). The next multiple of \( g \) is \( kG + kg = kG + x_{\min} \), which has even larger nonnegative coefficients than \( kG \). The next \( k - 1 \) multiples of \( g \) consequently also have nonnegative coefficients until we get to \( kG + 2x_{\min} \), and so on.

Note that the coefficients are not necessarily unique (all the \( x_i \) could be identical), but we have shown that there is at least one set of nonnegative coefficients for all multiple of \( g \) at least equal to \( X \).

4. Optimal money changing problem is that for a given \( A \), find the nonnegative \( a_i \)'s that satisfy \( A = \sum_{i=1}^{n} a_i x_i \), and such that the sum of all \( a_i \)'s is minimal —that is, you use the smallest possible number of coins. Here is a greedy algorithm for solving this problem:

Order your denominations such that \( x_1 > x_2 > \cdots > x_n \). Then the greedy algorithm for this problem would be: Given \( A \), let \( a_1 \) be the largest integer such that \( a_1 x_1 \leq A \). If \( A - a_1 x_1 > 0 \), let \( a_2 \) be the largest integer such that \( a_2 x_2 \leq A - a_1 x_1 \). If you have nothing left over after doing this for \( i = 1, \cdots, n \), then \( A = \sum_{i=1}^{n} a_i x_i \).
5. Let us show that the greedy algorithm finds the optimum $a_i$'s in the case of the denominations 1, 5, 10, and 25. Here is a proof.

Since 1 divides 5 and 5 divides 10, it is clear that if we have a case in which the greedy algorithm would not find the optimal solution, it must involve 25, i.e. $A$ must be greater than 25. Assume the greedy algorithm does not find the optimal solution for $A$, $A > 25$. Then $A = \sum_{i=1}^{4} a_i x_i = \sum_{i=1}^{4} b_i x_i$ and $\sum_{i=1}^{4} a_i > \sum_{i=1}^{4} b_i$, where the $a_i$ were determined by the greedy algorithm and the $b_i$ are optimal in that $\sum_{i=1}^{4} b_i$ is minimal. W.l.o.g. $a_4 = b_4$ [since $a_4 \leq 4$ any change of the number of 1 cent coins must occur in 5 unit steps to give the same sum—this is obviously worse than changing $b_3$], in addition to that note that $a_3 \leq 1$.

By the above considerations we must have $a_1 > b_1$. Let $x := a_1 - b_1$. We have three cases to consider: $a_2 = b_2$, $a_2 > b_2$ and $a_2 < b_2$. If we set $y := a_2 - b_2$ then we can compute $b_3 = 5x + 2y + a_3$. Thus the number of coins changes by $\sum_{i=1}^{4} b_i - \sum_{i=1}^{4} a_i = 4x + y$. If we can show that this number is positive, this is a contradiction and we are done. In cases 1 and 2, $x$ and $y$ are $\geq 0$. Therefore $4x + y$ is clearly positive.

In case 3, $y$ is negative. But, as we have to ensure that $b_3 = 5x + 2y + a_3$ is $\geq 0$ and we know that $a_3$ is at most 1, we have $y \geq -\frac{5}{2}x - \frac{1}{2}$. Hence $4x + y \geq \frac{3}{2}x - \frac{1}{2}$ and it is again positive.

6. You can extend this problem and ask “What are good necessary and sufficient conditions on a currency such that the greedy algorithm always gives the minimum amount of coins.” This problem is still open. Partial answers and a light-hearted discussion give:

