1. Let $A \in \mathbb{C}^{n\times n}$.

   (a) A scalar λ is an *eigenvalue* of an $n \times n$ A and a nonzero vector $x \in \mathbb{C}^n$ is a corresponding *(right) eigenvector* if
   $$Ax = \lambda x.$$

   (b) A nonzero vector y is called a *left eigenvector* if
   $$y^H A = \lambda y^H.$$

   (c) The set of all eigenvalues of A, denoted as $\lambda(A)$, is called the *spectrum* of A.

   (d) The *characteristic polynomial* of A is a polynomial of degree n, and defined as
   $$p(\lambda) = \det(\lambda I - A).$$

2. The following is a list of properties straightforwardly from above definitions:

   (a) λ is A's eigenvalue $\Leftrightarrow$ $\lambda I - A$ is singular $\Leftrightarrow$ $\det(\lambda I - A) = 0 \Leftrightarrow p(\lambda) = 0$.

   (b) There is at least one eigenvector x associated with A's eigenvalue λ.

   (c) Suppose A is real. λ is A's eigenvalue $\Leftrightarrow$ conjugate $\bar{\lambda}$ is also A's eigenvalue.

   (d) A is singular $\Leftrightarrow$ 0 is A's eigenvalue.

   (e) If A is upper (or lower) triangular, then its eigenvalues consist of its diagonal entries. (*Question: what if A is a block upper (or lower) triangular matrix ?*).

3. Schur decomposition. Let A be of order n. Then there is an $n\times n$ unitary matrix U (i.e., $U^H U = I$) such that
   $$A = UTU^H,$$
   where T is upper triangular and the diagonal elements of T are the eigenvalues of A.

4. When A is Hermitian, i.e., $A^H = A$, then by Schur decomposition, we know that there exist an unitary matrix U such that
   $$A = U\Lambda U^H,$$
   where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Furthermore, all eigenvalues λ_i are real.

   This result is also known as Spectral Theorem, considered a crowning result of linear algebra.

5. $A \in \mathbb{C}^{n\times n}$ is *simple* if it has n linearly independent eigenvectors; otherwise it is *defective*.

   Examples.

   (a) I and any diagonal matrices is simple. $e_1, e_2, \ldots, e_n$ *are* n linearly independent eigenvectors.

   (b) $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ is simple. It has two different eigenvalues -1 and 5, it has 2 linearly independent eigenvectors: $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

   (c) If $A \in \mathbf{C}^{n\times n}$ has n different eigenvalues, then A is simple.

(d) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is defective. It has two repeated eigenvalues 2, but only one eigenvector $e_1 = (1, 0)^T$.

6. *Eigenvalue decomposition*

   $A \in \mathbb{C}^{n \times n}$ is simple if and only if there exisits a nonsingular matrix $X \in \mathbf{C}^{n \times n}$ such that

$$A = X \Lambda X^{-1},$$

   where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. In this case, λ_i are eigenvalues, and the columns of X are eigenvectors, and A is called *diagonalizable*).

7. Two $n \times n$ matrices A and B are *similar* if there is an $n \times n$ non-singular matrix P such that $B = P^{-1}AP$. We also say A is *similar* to B, and likewise B is similar to A; P is a *similarity transformation*. A is *unitarily similar* to B if P is unitary.

   PROPOSITION. Suppose that A and B are similar: $B = P^{-1}AP$.

   (a) A and B have the same eigenvalues. In fact $p_A(\lambda) \equiv p_B(\lambda)$.

   (b) $Ax = \lambda x \Rightarrow B(P^{-1}x) = \lambda(P^{-1}x)$.

   (c) $Bw = \lambda w \Rightarrow A(Pw) = \lambda(Pw)$.