ECS130 Homework Assignment 2 **Due: 4:00pm, January 30, 2019**

1. (Exercise 4.8) Suppose $A\mathbb{R}^{n\times n}$ is tridiagonal

$$A = \begin{bmatrix} v_1 & w_1 & & & & \\ u_2 & v_2 & w_2 & & & \\ & u_3 & v_3 & w_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & u_{n-1} & v_{n-1} & w_{n-1} \\ & & & & u_n & v_n \end{bmatrix},$$

   and A is diagonal dominant, meaning that $|v_i| > |u_i|+|w_i|$ for all i. Show that in this case, the linear system $Ax = b$ can be solved in $O(n)$ time. *This is the well-known Thomas algorithm in computer graphics and other fields. Hint: compute the LU factorization of A, and then solve the triangular systems, find the recursions and show each recursion takes $O(n)$.*

2. (Exercise 4.2) Show that the matrix D introduced in §4.2.1 is symmetric positive definite.

3. (a) Show that a matrix A is *symmetric positive definite (spd)* if and only if there is a lower triangular matrix L such that $A = LL^T$, called Cholesky decomposition of A.

   (b) Write a program, called `myCholeksly`, to compute the Cholesky decomposition of an $n\times n$ spd matrix A. *Hint: see Figure 4.6*

   (c) A more general version of Cholesky decomposition that does not require the comptation of square roots is the LDLT decomposition. Here are your tasks for computing the LDLT decomposition.

   (c1) Show that if A is symmetric and admits an LU factorization (without pivtoing). Show that A can be factored $A = LDL^T$, where D is diagonal and L is lower triangular. (*Hint: take $D \equiv UL^{-T}$ and show that D is diagonal.*)

   (c2) Modify your program `myCholesky` to compute the LDLT decomposition of a spd matrix A without using any square root operations.

   (d) Check the correctness of your program for the matrices $A = (a_{ij})$ with $a_{ij} = \frac{1}{i+j-1}$ with $n = 3, 4, 5$.

   For the homework submission, include a copy of your program and the output of matrices L and D by your programs.

4. Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$ and assume that $A \in \mathbb{R}^{n\times n}$, show that if the rank of A is n, then $\|x\|_A \equiv \|Ax\|$ is a vector norm on $\mathbb{R}^n$.

5. Assume the matrix norm $\|A\|$ for $A \in \mathbb{R}^{n\times n}$ is induced by a vector norm $\|v\| \in \mathbb{R}^n$.

   (a) For $A \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^n$, show that $\|Av\| \le \|A\|\|v\|$.

   (b) For $A, B \in \mathbb{R}^{n\times n}$, show that $\|A + B\| \le \|A\| + \|B\|$.

   (c) For $A, B \in \mathbb{R}^{n\times n}$, show that $\|AB\| \le \|A\|\|B\|$.

6. If $Ax = b$ and $A\widehat{x} = \widehat{b}$, show that

$$\frac{\|x - \widehat{x}\|}{\|x\|} \le \kappa \frac{\|b - \widehat{b}\|}{\|b\|},$$

   where $\kappa = \|A\|\|A^{-1}\|$ is the condition number of $A \in \mathbb{R}^{n\times n}$. By this result, we learn that the condition number κ indicates the sensitivity of the linear system to the perturbation of b.