Linear system: $\overset{m\times n}{A}x = b$

1. Solvability

a) cases: 1)* no solution. e.g. $\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}$

2)* uniq. sol. $\begin{pmatrix}3 & 2\\ -4 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}6\\ 7\end{pmatrix} \rightarrow \overset{\text{unique sol.}}{\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-8/11\\ 4/11\end{pmatrix}}$

3)* many sol. $\begin{pmatrix}2 & 1 & 4\\ 3 & 2 & 6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}7\\ 11\end{pmatrix}$ $\rightarrow \begin{pmatrix}x_1\\ y_1\\ z_1\end{pmatrix}=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$ $\rightarrow \begin{pmatrix}x_2\\ y_2\\ z_2\end{pmatrix}=\begin{pmatrix}-1\\ 1\\ 2\end{pmatrix}$

$\rightarrow$ In fact, for any $t \in \mathbb{R}$ $t\begin{pmatrix}x_1\\ \vdots\end{pmatrix}+(1-t)\begin{pmatrix}x_2\\ \vdots\end{pmatrix}$ is a sol.

b) depending on "on b" : $Ax=b$ is solvable if b is in the column space of A

↳ e.g.: $\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$

c) depending on "on A" : $A = {}_m\overset{n}{\boxed{}}{}_m$:

- since $n \geq m$, cols of A must be linear independent
  i.e. $\exists\, x_0$, such that $Ax_0 = 0$
- if x is a sol. of $Ax=b$,
  then $A(x+\alpha x_0) = 0$

$\Rightarrow$ "wide" linear system admits many sols

$A = {}_m\overset{n}{\boxed{\rule{0pt}{2em}}}$ since $n<m$, cols of A cannot span $\underbrace{\mathbb{R}^m}_{\text{the larger-dimensional}}$

there exists b_0, such that $\underline{Ax=b_0}$ is not solvable.

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d) Assumption for now: $A = {}_m\overset{n}{\boxed{\rule{0pt}{1.5em}}}$, $m=n$ "Square"

↳ nonsingular $\Rightarrow A^{-1}$ exists

↳ unique sol. $\underline{x = A^{-1}\cdot b}$

2. GE ~~in action.~~ "ad hoc" way"

$$y - z = -1$$
$$3x - y + z = 4$$
$$x + y - 2z = -3$$

$$\longrightarrow \quad \begin{array}{ccccc} A & & \vec{x} & = & b \\ \begin{pmatrix} 0 & 1 & -1 \\ 3 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix} & & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & = & \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix} \end{array}$$

$\downarrow$ GE – "ad hoc"

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & -1 \\ 3 & -1 & 1 & 4 \\ 1 & 1 & -2 & -3 \end{array}\right)$$

$\downarrow$ ~~"ad hoc"~~ permutation $P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & -2 & -3 \end{array}\right)$$

$\downarrow$ row-scaling $S_1 = \begin{pmatrix} 1/3 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -1/3 & 1/3 & 4/3 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & -2 & -3 \end{array}\right)$$

$\downarrow$ elimination $\leftarrow M_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -1/3 & 1/3 & 4/3 \\ 0 & 1 & -1 & -1 \\ 0 & 4/3 & -7/3 & -13/3 \end{array}\right)$$

$\downarrow$ elimination $M_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & -\frac{4}{3} & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -1/3 & 1/3 & 4/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -3 \end{array}\right)$$

$\downarrow$ scaling $S_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -1/3 & 1/3 & 4/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$\longleftarrow$ elimination M_3 $\quad M_3 = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -1/3 & 1/3 & 4/3 \\ & 1 & 0 & 2 \\ & & 1 & 3 \end{array}\right)$$

$\uparrow$ elimination $M_4 = \begin{pmatrix} 1 & & -1/3 \\ & 1 & \\ & & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -1/3 & 0 & 1/3 \\ & 1 & 0 & 2 \\ & & 1 & 3 \end{array}\right)$$

$\uparrow$ elimination $M_5 = \begin{pmatrix} 1 & 1/3 & \\ & 1 & \\ & & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ & 1 & 0 & 2 \\ & & 1 & 3 \end{array}\right)$$

$\uparrow$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

3. Elementary matrix operations

1) Permutation.

$$P_\sigma = \begin{pmatrix} e^T_{\sigma(1)} \\ \vdots \end{pmatrix} \longrightarrow P_\sigma^T P_\sigma = I \longrightarrow P_\sigma^{-1} = P_\sigma^T$$

$$\text{e.g.} \quad I_{3\times3} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \xrightarrow[\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}]{} P_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

2) Row-scaling

$$\underset{a = (a_1, a_2, \cdots, a_m)}{S_a} = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_m \end{pmatrix} \xrightarrow[\substack{\text{if } a_i \neq 0 \\ \text{for any } i}]{} S_a^{-1} = \begin{pmatrix} 1/a_1 & & \\ & \ddots & \\ & & 1/a_m \end{pmatrix}$$

3) Elimination.

$$M = I + c\, e_l e_k^T \xrightarrow[\substack{\text{verify that} \\ M^{-1} \cdot M = I}]{} M^{-1} = I - c\, e_l e_k^T$$

Assume $k < l$

$$MA = \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots \\ \vdots & & & \\ a_{k1} & \cdots & \cdots & \cdots \\ \vdots & & & \\ \underline{a_{l1} + c\, a_{k1}} & \cdots & \cdots & \cdots \\ \vdots & & & \end{pmatrix} \leftarrow l^{th} \text{ row}$$

4. GE ~ in action

↳ "No pivoting"

$$Ax=b \longrightarrow M_1 A x = M_1 b$$

$$\downarrow$$

$$\underline{M_2 (M_1 A)}\, x = \underline{M_2} M_1 b$$

$$\vdots$$

$$(\underline{M_k \cdots M_1 A})\, x = \underbrace{(M_k \cdots M_1)\, b}_{\tilde{b}}$$

$$\downarrow$$

$$\boxed{U x = \tilde{b}}$$

upper triangular

$$\downarrow$$

solve $\underline{\underline{x}}$ by Back-substitution

LU Factorization

$$\underline{M_k \cdots M_2 M_1 A} = U \Longrightarrow A = (M_k \cdots M_1)^{-1} U$$

$$A = \underline{M_1^{-1} \cdots M_{k-1}^{-1} M_k^{-1}} \cdot U$$

$$\equiv L \cdot U$$

5. The needs of pivoting — mathematically and numerically

↳ Handout

↳ Exercise 3.12 (page 63)