Column Space and QR — Chap. 5

1) The main task:

Given $A \in \mathbb{R}^{m \times n}$

Find an $m \times m$ orthogonal matrix Q and an $m \times n$ upper triangular matrix R

such that

$$A = Q \cdot R$$

It's call (full) QR factorization of A.

It's easy to see the column space of A ~~can be~~ is spanned by the orthonormal columns of Q.

2) A set of vectors $\{q_1, \cdots, q_k\}$ is called orthonormal

if

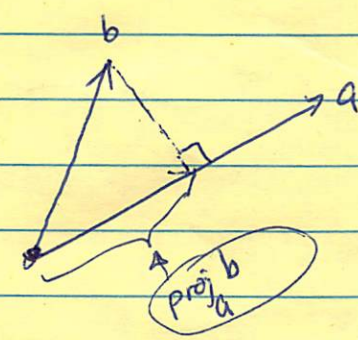
$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$Q = [q_1 \cdots q_k]$ is called an orthogonal matrix.

3) Projection

a) The projection of b onto a is the vector

$$\text{proj}_a b = \underbrace{\left(\frac{a^T b}{a^T a}\right)}_{c} \cdot a$$



where c is the minimizer of the least squares $\min_c \|b - c \cdot a\|_2$.

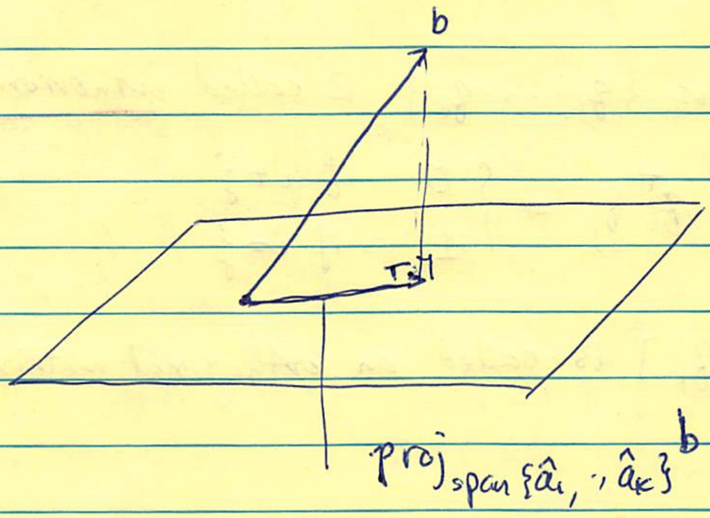
b) the projection of b onto span$\{\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_k\}$, where $\{\hat{a}_i\}$ are orthonormal:

$$\mathrm{proj}_{\mathrm{span}\{\hat{a}_1, \cdots, \hat{a}_k\}} b = [\hat{a}_1 \ \hat{a}_2 \cdots \hat{a}_k] \cdot c$$

where
$$c = \begin{pmatrix} \hat{a}_1^T b \\ \vdots \\ \hat{a}_k b \end{pmatrix}.$$

- In fact, c is the minimizer of the least squares

$$\min_c \| b - [\hat{a}_1 \cdots \hat{a}_k] \cdot c \|_2$$



4) Gram-Schmidt orthogonalization process = QR factorization (GS = QR)

Consider

$$A = [a_1\ a_2\ a_3\ \cdots] = [q_1, q_2, q_3, \cdots]\begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots \\ & r_{22} & r_{23} & \cdots \\ & & r_{33} & \cdots \\ & & & \ddots \end{bmatrix}$$

Then by the 1st col. of the eq.

$$a_1 = q_1 r_{11}$$

Therefore $\quad r_{11} = \|a_1\|_2, \quad q_1 = a_1 / r_{11}$

By the 2nd col.

$$a_2 = q_1 r_{12} + q_2 r_{22}$$

$$q_2 r_{22} = a_2 - q_1 r_{12} \equiv w_2$$

Therefore: $\quad r_{12} = q_1^T a_2.$

$$r_{22} = \|w_2\|_2, \quad q_2 = w_2 / \|w_2\|_2$$

By the 3rd col.

$$a_3 = q_1 r_{13} + q_2 r_{23} + q_3 r_{33}$$

$$q_3 r_{33} = a_3 - q_1 r_{13} - q_2 r_{23} \equiv w_3$$

Therefore $\quad r_{13} = q_1^T a_3, \quad r_{23} = q_2^T a_3$

$$r_{33} = \|w_3\|_2, \quad q_3 = w_3 / \|w_3\|_2$$

The procedure can be repeated until we find the desired QR factorization of A.

An important observation:

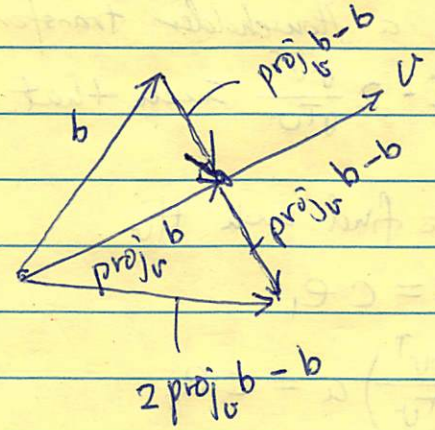
$$w_2 = a_2 - q_1 r_{12} \equiv a_2 - \mathrm{proj}_{q_1} a_2$$

$$w_3 = a_3 - (q_1 r_{13} + q_2 r_{23}) \equiv a_3 - \mathrm{proj}_{\mathrm{span}\{q_1, q_2\}} a_3$$

This indicates the connection between the QR (= GS) with the concept of projection — what heavily exploited in the textbook (section 5.4).

5) Householder Reflection/transformation.

~~Ho~~ How to find the reflection of a vector b over v?



By the above illustration, the Reflecting vector of b is

$$2\,\underline{proj_v}\,b - b = 2\,\frac{v^T b}{v^T v}\,v - b$$

$$= 2\,v \cdot \frac{v^T b}{v^T v} - b$$

$$= -\underbrace{\left(I - 2\,\frac{v v^T}{v^T v}\right)}\,b$$

$$\equiv -H_v \cdot b$$

$$H_v = I - 2\,\frac{v v^T}{v^T v}$$

is called Householder Reflector/transformation.

6) QR factorization by Householder transformation

Basic tool: given any vector $a \in \mathbb{R}^n$,
there is a Householder transformation
$H_v = I - 2\frac{vv^T}{v^Tv}$ such that $H_v a = ce_1$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Next we show how to find such H_v.

$$H_v a = c e_1$$

$$\left(I - 2\frac{vv^T}{v^Tv}\right) a = c e_1$$

$$v = (a - ce_1) \cdot \frac{v^Tv}{2v^Ta} \qquad (*)$$

Since scaling v does not affect H_v,

we can select

$$v = a - ce_1$$

By the selection of v, from $(*)$, we have

$$1 = \frac{v^Tv}{2v^Ta}$$

$$1 = \frac{(a-ce_1)^T(a-ce_1)}{2(a-ce_1)^Ta}$$

solve for c: $\quad \underline{\underline{c = \pm \|a\|_2}}$

Next, we use a simple numerical example to illustrate how to use the "Basic tool" to compute the QR factorization. The general procedure is in Fig. 5.9

7) Example to use the Householder transformation to compute the QR factorization

Consider
$$A = \begin{pmatrix} 2 & -1 & 5 \\ 2 & 1 & 2 \\ 1 & 0 & -2 \end{pmatrix}$$

the first col. of $A = a_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $c_1 = \|a_1\|_2 = 3$

$$v_1 = a_1 - c_1 e_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Then
$$H_{v_1} A = \left(I - 2\frac{v_1 v_1^T}{v_1^T v_1}\right) A = A - 2\frac{v_1 (v_1^T A)}{v_1^T v_1}$$

$$= \begin{pmatrix} 3 & 0 & 4 \\ 0 & -1 & 4 \\ 0 & -1 & -1 \end{pmatrix}$$

Now ~~for~~ the second column is : $\begin{pmatrix} -1 \\ -1 \end{pmatrix} \equiv \hat{a}_2$

the vector from the diagonal entry below in

$c_2 = \pm\|\hat{a}_2\|_2 = \pm\sqrt{2}$, take $c_2 = -\sqrt{2}$ (why not + ?)

$$v_2 = \begin{pmatrix} 0 \\ \hat{a}_2 \end{pmatrix} - c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1+\sqrt{2} \\ -1 \end{pmatrix}$$

Then we have
$$H_{v_2}(H_{v_1} A) = H_{v_2} \cdot \begin{pmatrix} 3 & 0 & 4 \\ 0 & -1 & 4 \\ 0 & -1 & -1 \end{pmatrix}$$

$$= \left(I - 2\frac{v_2 v_2^T}{v_2^T v_2}\right)\begin{pmatrix} 3 & 0 & 4 \\ 0 & -1 & 4 \\ 0 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 4 \\ 0 & -\sqrt{2} & 3/\sqrt{2} \\ 0 & 0 & 5/\sqrt{2} \end{pmatrix} \equiv R$$

From $\quad H_{v_2} H_{v_1} A = R$

we have $\quad A = H_{v_1}^{-1} H_{v_2}^{-1} R$ property of Householder $H_v^{-1} = H_v$

$= H_{v_1} H_{v_2} \cdot R$

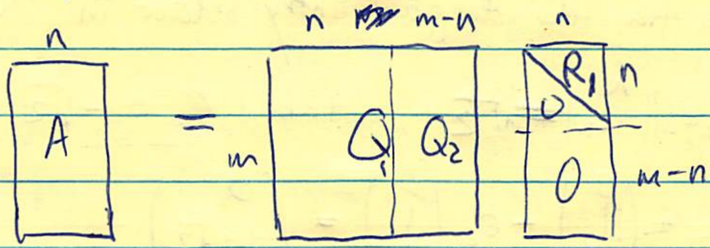
$= Q \cdot R$

specifically, $H_{v_1} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{pmatrix}$

$H_{v_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

8) Reduced QR / Compact QR

$\underset{m\times n}{A} =$



$= \underset{m\times n}{Q_1} \underset{n\times n}{R_1}$ ← Reduced QR also known as compact QR

Matlab: [Q, R] = qr(A)

vs. [Q, R] = qr(A, 0).