ECS130

Eigenvectors – Chapter 6

February 1, 2019

Eigenvalue problem

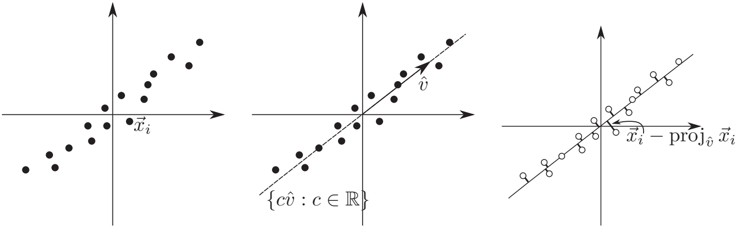
For a given $A \in \mathbb{C}^{m \times n}$, *find* $0 \neq x \in \mathbb{C}^n$ *and* $\lambda \in \mathbb{C}$, *such that*

$$Ax = \lambda x.$$

- x is called an eigenvector
- λ is called an eigenvalue
- (λ, x) is called an eigenpair

Motivation

Principal Component Analysis (PCA)



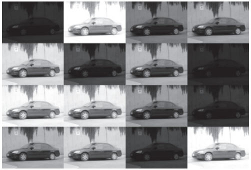
(a) Input data (b) Principal axis (c) Projection error

$$\text{minimize}_v \quad \sum_i \|x_i - \text{proj}_v x_i\|_2$$

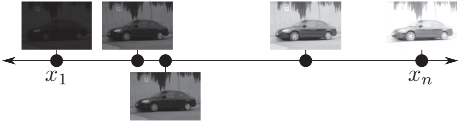
$$\text{subject to} \quad \|v\|_2 = 1$$

Motivation

Spectral Embedding



(a) Database of photos



(b) Spectral embedding

$$\begin{aligned} \text{minimize}_x \quad & E(x) = \sum_{i,j} w_{ij}(x_i - x_j)^2 \\ \text{subject to} \quad & x^T \mathbf{1} = 0 \\ & \|x\|_2 = 1, \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n)^T$.

Eigenvalues and eigenvectors

Let $A \in \mathbb{C}^{n \times n}$.

1. A scalar λ is an *eigenvalue* of an $n \times n$ A and a nonzero vector $x \in \mathbb{C}^n$ is a corresponding *(right) eigenvector* if

$$Ax = \lambda x.$$

   A nonzero vector y is called a *left eigenvector* if

$$y^H A = \lambda y^H.$$

2. The set $\lambda(A) = \{\text{all eigenvalues of } A\}$ is called the *spectrum* of A.
3. The *characteristic polynomial* of A is a polynomial of degree n:

$$p(\lambda) = \det(\lambda I - A).$$

Properties

The following is a list of properties straightforwardly from above definitions:

1. λ is A's eigenvalue $\Leftrightarrow \lambda I - A$ is singular $\Leftrightarrow$ $\det(\lambda I - A) = 0 \Leftrightarrow p(\lambda) = 0$.
2. There is at least one eigenvector x associated with A's eigenvalue λ.
3. Suppose A is real. λ is A's eigenvalue $\Leftrightarrow$ conjugate $\bar{\lambda}$ is also A's eigenvalue.
4. A is singular $\Leftrightarrow 0$ is A's eigenvalue.
5. If A is upper (or lower) triangular, then its eigenvalues consist of its diagonal entries.

Schur decomposition

Let A be of order n. Then there is an $n \times n$ unitary matrix U (i.e., $U^H U = I$) such that

$$A = UTU^H,$$

where T is upper triangular.

By the decomposition, we know that the diagonal elements of T are the eigenvalues of A.

Spectral Theorem

If A *is Hermitian, i.e.,* $A^H = A$*, then by Schur decomposition, we know that there exist an unitary matrix* U *such that*

$$A = U\Lambda U^H,$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$*. Furthermore, all eigenvalues* λ_i *are real.*

Spectral theorem is considered a *crowning result* of linear algebra.

Simple and defective matrices

$A \in \mathbb{C}^{n\times n}$ is *simple* if it has n linearly independent eigenvectors; otherwise it is *defective*.

Examples.

1. I and any diagonal matrices is simple. $e_1, e_2, \ldots, e_n$ *are* n linearly independent eigenvectors.
2. $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ is simple. It has two different eigenvalues -1 and 5, it has 2 linearly independent eigenvectors: $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
3. If $A \in \mathbf{C}^{n\times n}$ has n different eigenvalues, then A is simple.
4. $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is defective. It has two repeated eigenvalues 2, but only one eigenvector $e_1 = (1, 0)^T$.

Eigenvalue decomposition

$A \in \mathbb{C}^{n\times n}$ is **simple** *if and only if there exisits a nonsingular matrix $X \in \mathbf{C}^{n\times n}$ such that*

$$A = X\Lambda X^{-1},$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

In this case, $\{\lambda_i\}$ are eigenvalues, and columns of X are eigenvectors, and A is called *diagonalizable*.

Similarity transformation

- $n \times n$ matrices A and B are *similar* if there is an $n \times n$ non-singular matrix P such that $B = P^{-1}AP$.
- We also say A is *similar* to B, and likewise B is similar to A;
- P is a *similarity transformation*. A is *unitarily similar* to B if P is unitary.
- **Properties.** Suppose that A and B are similar:

$$B = P^{-1}AP.$$

  1. A and B have the same eigenvalues. In fact $p_A(\lambda) \equiv p_B(\lambda)$.
  2. $Ax = \lambda x \Rightarrow B(P^{-1}x) = \lambda(P^{-1}x)$.
  3. $Bw = \lambda w \Rightarrow A(Pw) = \lambda(Pw)$.