ECS130

Eigenvectors – Chapter 6

February 4, 2019

Power method

- Power iteration:

  Given an initial vector u_0,
  $i = 0$
  repeat
    $t_{i+1} = Au_i$
    $u_{i+1} = t_{i+1}/\|t_{i+1}\|_2$ (approximate eigenvector)
    $\theta_{i+1} = u_{i+1}^H A u_{i+1}$ (approximate eigenvalue)
    $i = i + 1$
  until convergence

- Simple stopping criterion: $|\theta_{i+1} - \theta_i| \leq tol \cdot |\theta_i|$.

Power method

Example: Let

$$A = \begin{bmatrix} -261 & 209 & -49 \\ -530 & 422 & -98 \\ -800 & 631 & -144 \end{bmatrix}.$$

and $\lambda(A) = \{10, 4, 3\}$.

Let $u_0 = (1, 0, 0)^T$, then

i	1	2	3	$\cdots$	10
θ_i	994.49	13.0606	10.07191	$\cdots$	10.0002

Power method

Convergence analysis: Assume that A is diagonalizable, i.e.,

$$A = X\Lambda X^{-1}$$

with $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$. Then, we can show that

- $u_i = \frac{A^i u_0}{\|A^i u_0\|} \to x_1/\|x_1\|$, where $x_1 = Xe_1$ as $i \to \infty$.
- $\theta_i \to \lambda_1$ as $i \to \infty$.
- The convergence rate depends on $\frac{|\lambda_2|}{|\lambda_1|}$.

Therefore, if $\frac{|\lambda_2|}{|\lambda_1|}$ is close to 1, then the power method could be very slow convergent or doesn't converge at all.

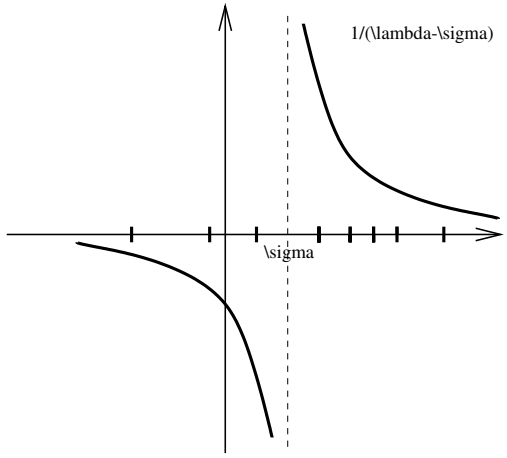
Inverse iteration

Purposes:

- Overcome the drawbacks of the power method (slow convergence)
- find an eigenvalue closest to a particular given number (called *shift*): σ

Observation: if λ is an eigenvalue of A, then

- $\lambda - \sigma$ is an eigenvalue of $A - \sigma I$,
- $\frac{1}{\lambda - \sigma}$ is an eigenvalue of $(A - \sigma I)^{-1}$.



Inverse iteration

Given an initial vector u_0 and a shift σ
$i = 0$
repeat
  solve $(A - \sigma I)t_{i+1} = u_i$ for t_{i+1}
  $u_{i+1} = t_{i+1}/\|t_{i+1}\|_2$ (approximate eigenvector)
  $\theta_{i+1} = u_{i+1}^H A u_{i+1}$ (approximate eigenvalue)
  $i = i + 1$
until convergence

If we change the shift σ in each iteration:

$$\sigma = u_{i+1}^H A u_{i+1},$$

we have a so-called *Rayleigh quotient iteration*.

Inverse iteration

Convergence analysis: Assume $A = X\Lambda X^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and λ_k is the eigenvalue cloest to the shift σ. It can be shown that

- $u_i \to x_k/\|x_k\|$ as $i \to \infty$, where $x_k = Xe_k$
- θ_i converges to λ_k $i \to \infty$.
- Convergence rate depends on $\max_{j \neq k} \frac{|\lambda_k - \sigma|}{|\lambda_j - \sigma|}$.

Inverse iteration

1. Advantages:
   - the ability to converge to any desired eigenvalue nearest to the shift σ;
   - typically converges very quickly, and is particularly effective when we have a good approximation to an eigenvalue and want only its corresponding eigenvector.
2. Drawbacks:
   - expensive in general: solving linear systems $(A - \sigma I)t_{i+1} = u_i$ for u_{i+1}, which could be very expensive for large matrices
   - Only compute one eigenpair.

Orthogonal (subspace/simultaneous) iteration

- Purpose: compute p eigenvalues (and eigenvectors) at a time.
- Orthogonal iteration

  Given an initial $n \times p$ orthogonal matrix Z_0
  $i = 0$
  repeat
    $Y_{i+1} = AZ_i$
    $Y_{i+1} = Z_{i+1}R_{i+1}$ (QR decomposition)
    $i = i + 1$
  until convergence

- The use of QR decomposition keeps the vectors spanning $\text{span}\{A^i Z_0\}$ of full rank.

Orthogonal (subspace/simultaneous) iteration

Example: Let $Z_0 = [e_1, e_2, e_3]$ and

```
A =
-0.4326  1.1892 -0.5883 -0.0956 -0.6918 -0.3999
-1.6656 -0.0376  2.1832 -0.8323  0.8580  0.6900
 0.1253  0.3273 -0.1364  0.2944  1.2540  0.8156
 0.2877  0.1746  0.1139 -1.3362 -1.5937  0.7119
-1.1465 -0.1867  1.0668  0.7143 -1.4410  1.2902
 1.1909  0.7258  0.0593  1.6236  0.5711  0.6686

Eigvals of A = -2.1659+-0.5560i, 2.1493, 0.2111+-1.9014i, -0.9548
```

Eigenvalues of $Z_i^T A Z_i$ for $i = 10, 30, 70$:

```
i=10: Eigvals of Z'_10*A*Z_10: -1.4383+-0.3479i, 2.1500

i=30: Eigvals of Z'_30*A*Z_30: -2.1592+-0.5494i, 2.1118

i=70: Eigvals of Z'_70*A*Z_70: -2.1659+-0.5560i, 2.1493
```

Orthogonal (subspace/simultaneous) iteration

Convergence results:

- under mild conditions, Z_i converges to a subspace spanned by the first p eigenvectors corresponding to the p dominant eigenvalues, where

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_p| > |\lambda_{p+1}| \geq \cdots \geq |\lambda_n|.$$

- If we let $B_i = Z_i^T A Z_i$, then

$$\|AZ_i - Z_i B_i\| \to 0 \quad \text{as} \quad i \to \infty$$

  and eigenvalues of B_i approximate the dominant eigenvalues of A.
- Convergence rate depends on $|\lambda_{p+1}|/|\lambda_p|$.

QR iteration

- Goal: reorganize orthogonal iteration to incorporate shifting and inverting as in the inverse iteration. This will make it more efficient and eliminate the assumption that eigenvalues differ in magnitude.
- QR iteration

  $A_0 = A$
  $i = 0$
  repeat
    $A_i = Q_i R_i$ (QR decomposition)
    $A_{i+1} = R_i Q_i$
    $i = i + 1$
  until convergence

QR iteration

Properties:

- Observe that $A_{i+1} = R_i Q_i = Q_i^T Q_i R_i Q_i = Q_i^T A_i Q_i$. Therefore it performs an orthogonal similarity transformation at each iteration.
- A_{i+1} is orthogonally similar to $A_0 = A$:

$$A_{i+1} = (Q_0 Q_1 \cdots Q_{i-1} Q_i)^T A (Q_0 Q_1 \cdots Q_{i-1} Q_i).$$

Therefore A_{i+1} and A have same eigenvalues. (why? homework)

QR iteration

Example. The same test matrix as before.

After 10 iterations:

```
A_10 =
   -1.6994    0.2201   -0.8787   -1.4292   -0.3847    0.0112
   -0.0007    1.1325    1.2186    1.2245   -0.0867    0.0648
    0.2637    1.9636   -0.1598   -2.3959    0.8136   -0.4311
    0.0364   -0.2346   -0.5527   -0.4393   -1.9263    1.2496
    0.4290    1.3482   -1.1484    0.6121   -0.5937    0.2416
    0.0003    0.0013   -0.0003   -0.0011    0.0014   -0.9554
```

QR iteration

Example. The same test matrix as before.

After 30 iterations:

```
A_30 =
   -2.4055   -1.0586    1.3420   -0.0991    1.1210    0.1720
    0.0517    0.9645    1.6519   -0.8512   -0.7215    0.7654
   -0.2248    1.9947   -0.7656   -1.1876   -0.2736   -0.1552
   -0.0029   -0.0263   -0.0682    0.1381   -2.3094    0.6765
   -0.0147    0.0808   -0.0569    1.5462    0.3082   -0.8476
    0.0000    0.0000    0.0000    0.0000    0.0000   -0.9548
```

- From the last rwo of A_{30}, we can conclude "-0.9548 is an eigenvalue of A".
- The subsequent QR iterations are performed on the leading 4×4 submatrices to find the rest of eigenvalues.

QR iteration with shifts $\Rightarrow$ QR Algorithm

1. Purpose: accelerate the convergence of QR iteration by using shifts
2. QR Iteration with shifts

   $A_0 = A$; $i = 0$
   repeat
      Choose a shift σ_i
      $A_i - \sigma_i I = Q_i R_i$ (QR decomposition)
      $A_{i+1} = R_i Q_i + \sigma_i I$
      $i = i + 1$
   until convergence

QR iteration with shifts $\Rightarrow$ QR Algorithm

Property:

- A_i and A_{i+1} are orthogonally similar: $A_{i+1} = Q_i^T A_i Q_i$.
- Therefore, A_{i+1} and A are orthogonally similar, and A_{i+1} and A have the same eigenvalues.

why? homework!

QR iteration with shifts $\Rightarrow$ QR Algorithm

How to choose the shifts σ_i?

- If σ_i is an exact eigenvalue of A, then it can be shown that

$$A_{i+1} = R_i Q_i + \sigma_i I = \begin{bmatrix} \widehat{A} & a \\ 0 & \sigma_i \end{bmatrix}.$$

  This means that the algorithm converges in one iteration. If more eigenvalues are wanted, we can apply the algorithm again to the $(n-1) \times (n-1)$ matrix $\widehat{A}$.

- In practice, pick $\sigma_i = A_i(n,n)$.
  Reason: observing that the convergence of the QR iteration (without a shift), the (n,n) entry of A_i usually converges to an eigenvalue of A first.

QR iteration with shifts $\Rightarrow$ QR Algorithm

Example. The same test matrix as before. The following is the numerical result of QR iteration with a shift.

With the shift $\sigma_0 = 2.1493$, an "exact" eigenvalue of A, after one iteration:

```
A_1 =
   -1.4127    1.4420    1.0845   -0.6866   -0.1013   -0.2042
   -1.2949   -0.2334    1.4047   -1.3695    1.5274   -0.7062
    0.5473    0.1343   -0.7991   -0.6716    1.1585    0.0736
   -0.2630    0.0284    0.5440   -1.4616   -1.5892    0.9205
   -1.6063   -0.3898    0.3410    0.1623   -0.9576   -0.5795
    0.0000    0.0000    0.0000    0.0000    0.0000    2.1493
```

We observe that by the QR iteration converged in one-iteration.

QR iteration with shifts $\Rightarrow$ QR Algorithm

With the shifts $\sigma_i = A_i(n,n)$, after 7 iteration:

```
A_7 =
   -2.4302    2.0264   -0.2799   -0.2384    0.3210   -0.0526
   -0.1865   -1.4295   -1.3515    0.0812    0.8577   -0.0388
   -0.1087   -0.8991    0.4491    0.4890   -1.8463   -1.2034
   -0.0008    0.0511   -0.5997   -0.7839   -0.8088   -0.5188
   -0.0916   -0.8273    1.6940    0.0645   -0.6698   -0.0854
    0.0000    0.0000    0.0000    0.0000    0.0000    2.1493
```

We observe that by 7th iteration, from the last row, we have found an eigenvalue 2.1493 of A. The subsequent QR iterations with shift are then performed on the leading 4×4 submatrices to find the rest of eigenvalues.

QR iteration with shifts $\Rightarrow$ QR Algorithm

1. The QR decomposition takes $\mathcal{O}(n^3)$ flops. Even if the QR iteration took n iterations to converge, the overall cost will be $\mathcal{O}(n^4)$. This is too expensive.[1]
2. However, if the matrix is initially reduced to *upper Hessenberg form*, then the QR decomposition of a Hessenberg form costs $\mathcal{O}(n^2)$ flops. As a result, the overall cost of the algorithm is reduced to $\mathcal{O}(n^3)$. This is referred to as the **Hessenberg QR algorithm**.
3. The **Hessenberg QR algorithm** is the method of choice for dense eigenvalue problem today, and is considered as one of the top 10 algorithms invented in the 20th century.

[1] The complexity of algorithms for all standard matrix computation problems is at $\mathcal{O}(n^3)$.