1. Let $A \in \mathbb{C}^{n \times n}$.

(a) A scalar $\lambda$ is an eigenvalue of an $n \times n$ $A$ and a nonzero vector $x \in \mathbb{C}^n$ is a corresponding (right) eigenvector if

$$Ax = \lambda x.$$ 

(b) A nonzero vector $y$ is called a left eigenvector if

$$y^H A = \lambda y^H.$$ 

(c) The set of all eigenvalues of $A$, denoted as $\lambda(A)$, is called the spectrum of $A$.

(d) The characteristic polynomial of $A$ is a polynomial of degree $n$, and defined as

$$p(\lambda) = \det(\lambda I - A).$$

2. The following is a list of properties straightforwardly from above definitions:

(a) $\lambda$ is $A$’s eigenvalue $\iff \lambda I - A$ is singular $\iff \det(\lambda I - A) = 0 \iff p(\lambda) = 0$.

(b) There is at least one eigenvector $x$ associated with $A$’s eigenvalue $\lambda$.

(c) Suppose $A$ is real. $\lambda$ is $A$’s eigenvalue $\iff$ conjugate $\bar{\lambda}$ is also $A$’s eigenvalue.

(d) $A$ is singular $\iff 0$ is $A$’s eigenvalue.

(e) If $A$ is upper (or lower) triangular, then its eigenvalues consist of its diagonal entries.

(Question: what if $A$ is a block upper (or lower) triangular matrix ?).


Let $A$ be of order $n$. Then there is an $n \times n$ unitary matrix $U$ (i.e., $U^H U = I$) such that

$$A = U T U^H,$$

where $T$ is upper triangular and the diagonal elements of $T$ are the eigenvalues of $A$.

4. When $A$ is Hermitian, i.e., $A^H = A$, then by Schur decomposition, we know that there exist an unitary matrix $U$ such that

$$A = U \Lambda U^H,$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and all eigenvalues $\lambda_i$ are real.

5. $A \in \mathbb{C}^{n \times n}$ is simple if it has $n$ linearly independent eigenvectors; otherwise it is defective.

Examples.

(a) $I$ and any diagonal matrices is simple. $e_1, e_2, \ldots, e_n$ are $n$ linearly independent eigenvectors.

(b) $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ is simple. It has two different eigenvalues $-1$ and $5$, it has 2 linearly independent eigenvectors: $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
(c) If $A \in \mathbb{C}^{n \times n}$ has $n$ different eigenvalues, then $A$ is simple.

(d) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is defective. It has two repeated eigenvalues 2, but only one eigenvector $e_1 = (1, 0)^T$.

6. Eigenvalue decomposition

$A \in \mathbb{C}^{n \times n}$ is simple if and only if there exists a nonsingular matrix $X \in \mathbb{C}^{n \times n}$ such that

$$A = X\Lambda X^{-1},$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. In this case, $\lambda_i$ are eigenvalues, and the columns of $X$ are eigenvectors, and $A$ is called diagonalizable.

7. An invariant subspace of $A$ is a subspace $\mathcal{V}$ of $\mathbb{R}^n$, with the property that

$$v \in \mathcal{V} \text{ implies that } Av \in \mathcal{V}.$$  

We also write this as $A\mathcal{V} \subseteq \mathcal{V}$.

Examples.

(a) The simplest, one-dimensional invariant subspace is the set $\text{span}(x)$ of all scalar multiples of an eigenvector $x$.

(b) Let $x_1, x_2, \ldots, x_m$ be any set of independent eigenvectors with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then $\mathcal{X} = \text{span}\{x_1, x_2, \ldots, x_m\}$ is an invariant subspace.

PROPOSITION. Let $A$ be $n$-by-$n$, let $V = [v_1, v_2, \ldots, v_m]$ be any $n$-by-$m$ matrix with linearly independent columns, and let $\mathcal{V} = \text{span}(V)$, the $m$-dimensional space spanned by the columns of $V$. Then $\mathcal{V}$ is an invariant subspace if and only if there is an $m$-by-$m$ matrix $B$ such that

$$AV = VB.$$  

In this case, the $m$ eigenvalues of $B$ are also eigenvalues of $A$.

8. Two $n \times n$ matrices $A$ and $B$ are similar if there is an $n \times n$ non-singular matrix $P$ such that $B = P^{-1}AP$. We also say $A$ is similar to $B$, and likewise $B$ is similar to $A$; $P$ is a similarity transformation. $A$ is unitarily similar to $B$ if $P$ is unitary.

PROPOSITION. Suppose that $A$ and $B$ are similar: $B = P^{-1}AP$.

(a) $A$ and $B$ have the same eigenvalues. In fact $p_A(\lambda) \equiv p_B(\lambda)$.

(b) $Ax = \lambda x \Rightarrow B(P^{-1}x) = \lambda(P^{-1}x)$.

(c) $Bw = \lambda w \Rightarrow A(Pw) = \lambda(Pw)$. 

2