1. Overview

   - Study a way to describe the growth of functions in the limit – *asymptotic efficiency*
   - Focus on what's important (leading factor) by abstracting lower-order terms and constant factors
   - A way to compare "sizes" of functions

2. **Big-O notation**

   Let $f(x)$ and $g(x)$ be functions defined on the set (or subset) of real numbers. If there exists a real number k and a positive constant C such that

   $$|f(x)| \leq C|g(x)| \qquad \text{whenever} \quad x > k,$$

   then we say "$f(x)$ is big-O of $g(x)$", written

   $$f(x) = O(g(x)).$$

3. Examples:

   - $f(x) = x^2 + 2x + 1 = O(x^2)$
   - $1 + 2 + 3 + \cdots + n$ is $O(n^2)$.
   - $f(n) = n! = O(n^n)$
   - $g(n) = \log n! = O(n \log n)$.

   Note: C and k are not unique!

4. Theorem: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, \ldots, a_0$ are real numbers. Then $f(x) = O(x^n)$.

   Proof: If $x > 1$

   $$\begin{aligned} |f(x)| &\leq |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| \\ &\leq (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|)x^n \equiv Cx^n. \end{aligned}$$

5. Theorem: If $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$, then

   - $(f_1 + f_2)(x) = O(\max(|g_1(x)|, |g_2(x)|))$.
   - $(f_1 f_2)(x) = O(g_1(x)g_2(x))$.

6. Example: give a big O-notation estimate for $f(n) = 3n \log(n!) + (n^2 + 3) \log n$

7. An *ordering* of the growth of commonly used functions in computer science, namely if $f(n)$ comes before $g(n)$ in the order then $f(n) = O(g(n))$.

   $$\log n, \qquad n, \qquad n \log n, \qquad n^k, \qquad 2^n$$

8. **Big-Ω notation**

   Let $f(x)$ and $g(x)$ be functions defined on the set (or subset) of real numbers. If there exists a real number k and a positive constant C such that

   $$|f(x)| \geq C|g(x)| \qquad \text{whenever} \quad x > k,$$

   then we say "$f(x)$ is big-Omega of $g(x)$", written

   $$f(x) = \Omega(g(x)).$$

9. Example: $f(x) = 8x^3 + 5x^2 + 6$ is $\Omega(x^3)$.

10. **Big-Θ notation**

    Let $f(x)$ and $g(x)$ be functions defined on the set (or subset) of real numbers.

    $$f(x) = \Theta(g(x))$$

    if and only if

    $$f(x) = O(g(x)) \quad \text{and} \quad f(x) = \Omega(g(x)).$$

    reads: "$f(x)$ is big-Theta of $g(x)$", or $f(x)$ is of **(same) order** $g(x)$.

11. Example: $1 + 2 + \cdots + n$ is of order n^2, i.e.,

    $$1 + 2 + \cdots + n = \Theta(n^2).$$

12. Theorem: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, \ldots, a_0$ are real numbers and $a_n \neq 0$. Then $f(x)$ is of order x^n, i.e. $f(x) = \Theta(x^n)$.