II.3 Frequency Response Analysis via Algebraic Substructuring
Frequency response calculation

• Frequency response

\[ H(\omega) = l^T (K - \omega^2 M + i\omega D)^{-1} b \]
\[ = l^T [K - \omega^2 M + i\omega (\alpha K + \beta M)]^{-1} b \]
\[ \downarrow \sigma \text{(shift)} \]
\[ = l^T (\gamma_1 K^\sigma + \gamma_2 M)^{-1} b \]

• \( H(\omega) \)-projection onto the AS subspace \( \text{span}\{L^{-1}S_m\} \):

\[ H_m(\omega) = l_m^T (\gamma_1 K_m^\sigma + \gamma_2 M_m)^{-1} b_m = l^T p_m(\omega) \]

• Frequency response equation

\[ (\gamma_1 K_m^\sigma + \gamma_2 M_m) p_m(\omega) \equiv G_m(\omega)p_m(\omega) = b_m \]

• Global modes \( \Phi = (\phi) = [\Phi_l \ \Phi_n \ \Phi_r] \) of \( (K_m^\sigma, M_m) \)
Decomposition $p_m(\omega) = p_n(\omega) + p_t(\omega)$,
where $p_n(\omega) \in \text{span}\{\Phi_n\}$ and $p_t(\omega) \in \text{span}\{[\Phi_l \ \Phi_r]\}$.
Note that the orthogonal condition: $p_n^T(\omega)(K^\sigma_m, M_m)p_t^T(\omega) = 0$.

Phase 1: compute $p_n(\omega)$ directly

Let $p_n(\omega) = \Phi_n \eta_n(\omega)$ for some $\eta_n(\omega)$. Then

$$G_m(\omega) (\Phi_n \eta_n(\omega) + p_t(\omega)) = b_m$$

Pre-multiplying by $\Phi_n^T$ and apply the orthogonal condition, we have

$$\eta_n(\omega) = (\Phi_n^T G_m(\omega) \Phi_n)^{-1} \Phi_n^T b_m$$

Hence

$$p_n(\omega) = \Phi_n \eta_n(\omega) = \Phi_n (\gamma_1 \Theta_n^\sigma + \gamma_2 I)^{-1} \Phi_n^T b_m.$$ 

Phase 2: compute $p_t(\omega)$ using frequency sweeping iteration

Note that

$$G_m(\omega)p_t(\omega) = b_m - G_m(\omega)p_n(\omega)$$
It is anticipated $p_t(\omega)$ varies slowly with frequency, we can use an iterative refinement.

$$p_t^\ell(\omega) = p_t^{\ell-1}(\omega) + \Delta p_t^\ell(\omega)$$

with initial response $p_t^0(\omega)$ by (linear) extrapolation from previous frequency point.

Then apply the Gerlerkin projection approximation to solve

$$G_m(\omega)\Delta p_t^\ell(\omega) = r_m^{\ell-1}(\omega) \equiv b_m - G_m(\omega)p_m^\ell(\omega)$$

We have the following so-called frequency sweeping iteration:

$$p_t^\ell(\omega) = p_t^{\ell-1}(\omega) + \frac{1}{\gamma_1} \left[ (K_m^{-\sigma})^{-1} - \Phi_n(\Theta_n^{-\sigma})^{-1}\Phi_n^T \right] r_m^{\ell-1}(\omega)$$

- **ASFRA package** = ASEIG + FRA
  
  [Bennighof’98, ..., Ko& B.’08, Meerbergen and B.’09]
Convergence analysis of FS – cutoff values for $\Phi_n$

• Truncated modal residual

$$\Phi^T_t r_m^\ell(\omega) = -\frac{\gamma^2}{\gamma_1} \Theta_t^\sigma \Phi^T_t r_m^{\ell-1}(\omega).$$

• Contraction for the truncated mode $\Phi_t = (\phi_k)$:

$$\left| \frac{\phi_k^T r_m^\ell(\omega)}{\phi_k^T r_m^{\ell-1}(\omega)} \right| \leq \frac{d_{\text{max}}}{|\theta_k^\sigma|} \leq \xi < 1,$$

• Global cutoff values (to determine retained modes $\Phi_n$) eigenpairs of the original matrix pair $(K, M)$:

$$[\lambda_{\text{min}}, \lambda_{\text{max}}] = \left[ \sigma - \frac{d_{\text{max}}}{\xi}, \sigma + \frac{d_{\text{max}}}{\xi} \right]$$
Case study: checkerboard filter

- FE simulation of a prototype checkerboard MEMS resonator for a high-frequency bandpass filter, *e.g.*, *the surface acoustic wave devices in a cell phone*

The SEM picture of a fabricated device and FE models, courtesy of Bindel *et al.*

- \( N = 15258, \ [f_{\text{min}}, f_{\text{max}}] = [230, 250]\text{MHz}, \)
Case study: checkerboard filter – performance

- Accuracy of frequency responses:

- Performance

<table>
<thead>
<tr>
<th>m (AS subspace)</th>
<th>Direct</th>
<th>SIL</th>
<th>ASFRA(^0)</th>
<th>ASFRA</th>
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<tr>
<td>n (retained modes)</td>
<td>– 242</td>
<td>231</td>
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<td>Total FS iter</td>
<td>– 96</td>
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<td>Elapsed time (sec.)</td>
<td>1612.6 86.23</td>
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Further reading

For further detail, see the following paper and references therein, available on the class website:


A more recent paper is