Mathematics for Computer Science

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Eric Lehman
Google Inc.

F Thomson Leighton
Department of Mathematics
and the Computer Science and AI Laboratory,
Massachussetts Institute of Technology;
Akamai Technologies

Albert R Meyer
Department of Electrical Engineering and Computer Science
and the Computer Science and AI Laboratory,
Massachussetts Institute of Technology

6 Induction

Induction is a powerful method for showing a property is true for all nonnegative integers. Induction plays a central role in discrete mathematics and computer science, and in fact, its use is a defining characteristic of *discrete* —as opposed to *continuous* —mathematics. This chapter introduces two versions of induction — Ordinary and Strong —and explains why they work and how to use them in proofs. It also introduces the Invariant Principle, which is a version of induction specially adapted for reasoning about step-by-step processes.

6.1 Ordinary Induction

To understand how induction works, suppose there is a professor who brings to class a bottomless bag of assorted miniature candy bars. She offers to share the candy in the following way. First, she lines the students up in order. Next she states two rules:

1. The student at the beginning of the line gets a candy bar.
2. If a student gets a candy bar, then the following student in line also gets a candy bar.

Let's number the students by their order in line, starting the count with 0, as usual in computer science. Now we can understand the second rule as a short description of a whole sequence of statements:

- If student 0 gets a candy bar, then student 1 also gets one.
- If student 1 gets a candy bar, then student 2 also gets one.
- If student 2 gets a candy bar, then student 3 also gets one.

$$\vdots$$

Of course this sequence has a more concise mathematical description:

> If student n gets a candy bar, then student $n + 1$ gets a candy bar, for all nonnegative integers n.

So suppose you are student 17. By these rules, are you entitled to a miniature candy bar? Well, student 0 gets a candy bar by the first rule. Therefore, by the second rule, student 1 also gets one, which means student 2 gets one, which means student 3 gets one as well, and so on. By 17 applications of the professor's second rule, you get your candy bar! Of course the rules actually guarantee a candy bar to *every* student, no matter how far back in line they may be.

6.1.1 A Rule for Ordinary Induction

The reasoning that led us to conclude that every student gets a candy bar is essentially all there is to induction.

The Principle of Induction.

Let P be a predicate on nonnegative integers. If

- $P(0)$ is true, and
- $P(n)$ IMPLIES $P(n+1)$ for all nonnegative integers, n,

then

- $P(m)$ is true for all nonnegative integers, m.

Since we're going to consider several useful variants of induction in later sections, we'll refer to the induction method described above as *ordinary induction* when we need to distinguish it. Formulated as a proof rule as in Section 1.4.1, this would be

Rule. ***Induction Rule***

$$\frac{P(0), \quad \forall n \in \mathbb{N}.\, P(n) \text{ IMPLIES } P(n+1)}{\forall m \in \mathbb{N}.\, P(m)}$$

This general induction rule works for the same intuitive reason that all the students get candy bars, and we hope the explanation using candy bars makes it clear why the soundness of the ordinary induction can be taken for granted. In fact, the rule is so obvious that it's hard to see what more basic principle could be used to justify it.[1] What's not so obvious is how much mileage we get by using it.

[1] But see Section 6.3.

6.1.2 A Familiar Example

The formula (6.1) below for the sum of the nonnegative integers up to n is the kind of statement about all nonnegative integers to which induction applies directly. We already proved it (Theorem 2.2.1) using the Well Ordering Principle, but now we'll prove it using induction.

Theorem. *For all $n \in \mathbb{N}$,*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{6.1}$$

To use the Induction Principle to prove the Theorem, define predicate $P(n)$ to be the equation (6.1). Now the theorem can be restated as the claim that $P(n)$ is true for all $n \in \mathbb{N}$. This is great, because the induction principle lets us reach precisely that conclusion, provided we establish two simpler facts:

- $P(0)$ is true.
- For all $n \in \mathbb{N}$, $P(n)$ IMPLIES $P(n+1)$.

So now our job is reduced to proving these two statements. The first is true because $P(0)$ asserts that a sum of zero terms is equal to $0(0+1)/2 = 0$, which is true by definition.

The second statement is more complicated. But remember the basic plan from Section 1.5 for proving the validity of any implication: *assume* the statement on the left and then *prove* the statement on the right. In this case, we assume $P(n)$—namely, equation (6.1) —in order to prove $P(n+1)$, which is the equation

$$1 + 2 + 3 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}. \tag{6.2}$$

These two equations are quite similar; in fact, adding $(n+1)$ to both sides of equation (6.1) and simplifying the right side gives the equation (6.2):

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{(n+2)(n+1)}{2} \end{aligned}$$

Thus, if $P(n)$ is true, then so is $P(n+1)$. This argument is valid for every nonnegative integer n, so this establishes the second fact required by the induction principle. Therefore, the induction principle says that the predicate $P(m)$ is true for all nonnegative integers, m, so the theorem is proved.

6.1.3 A Template for Induction Proofs

The proof of equation (6.1) was relatively simple, but even the most complicated induction proof follows exactly the same template. There are five components:

1. **State that the proof uses induction.** This immediately conveys the overall structure of the proof, which helps your reader follow your argument.

2. **Define an appropriate predicate $P(n)$.** The predicate $P(n)$ is called the *induction hypothesis*. The eventual conclusion of the induction argument will be that $P(n)$ is true for all nonnegative n. Clearly stating the induction hypothesis is often the most important part of an induction proof, and omitting it is the largest source of confused proofs by students.

   In the simplest cases, the induction hypothesis can be lifted straight from the proposition you are trying to prove, as we did with equation (6.1). Sometimes the induction hypothesis will involve several variables, in which case you should indicate which variable serves as n.

3. **Prove that $P(0)$ is true.** This is usually easy, as in the example above. This part of the proof is called the *base case* or *basis step*.

4. **Prove that $P(n)$ implies $P(n+1)$ for every nonnegative integer n.** This is called the *inductive step*. The basic plan is always the same: assume that $P(n)$ is true and then use this assumption to prove that $P(n+1)$ is true. These two statements should be fairly similar, but bridging the gap may require some ingenuity. Whatever argument you give must be valid for every nonnegative integer n, since the goal is to prove the implications $P(0) \rightarrow P(1)$, $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, etc. all at once.

5. **Invoke induction.** Given these facts, the induction principle allows you to conclude that $P(n)$ is true for all nonnegative n. This is the logical capstone to the whole argument, but it is so standard that it's usual not to mention it explicitly.

Always be sure to explicitly label the *base case* and the *inductive step*. It will make your proofs clearer, and it will decrease the chance that you forget a key step (such as checking the base case).

6.1.4 A Clean Writeup

The proof of the Theorem given above is perfectly valid; however, it contains a lot of extraneous explanation that you won't usually see in induction proofs. The writeup below is closer to what you might see in print and should be prepared to produce yourself.

Revised proof of the Theorem. We use induction. The induction hypothesis, $P(n)$, will be equation (6.1).

Base case: $P(0)$ is true, because both sides of equation (6.1) equal zero when $n = 0$.

Inductive step: Assume that $P(n)$ is true, where n is any nonnegative integer. Then

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) && \text{(by induction hypothesis)} \\ &= \frac{(n+1)(n+2)}{2} && \text{(by simple algebra)} \end{aligned}$$

which proves $P(n+1)$.

So it follows by induction that $P(n)$ is true for all nonnegative n. ■

Induction was helpful for *proving the correctness* of this summation formula, but not helpful for *discovering* it in the first place. Tricks and methods for finding such formulas will be covered in Part III of the text.

6.1.5 A More Challenging Example

During the development of MIT's famous Stata Center, as costs rose further and further beyond budget, there were some radical fundraising ideas. One rumored plan was to install a big courtyard with dimensions $2^n \times 2^n$ with one of the central squares[2] occupied by a statue of a wealthy potential donor —who we will refer to as "Bill", for the purposes of preserving anonymity. The $n = 3$ case is shown in Figure 6.1.

A complication was that the building's unconventional architect, Frank Gehry, was alleged to require that only special L-shaped tiles (shown in Figure 6.2) be used for the courtyard. For $n = 2$, a courtyard meeting these constraints is shown in Figure 6.3. But what about for larger values of n? Is there a way to tile a $2^n \times 2^n$ courtyard with L-shaped tiles around a statue in the center? Let's try to prove that this is so.

Theorem 6.1.1. *For all $n \geq 0$ there exists a tiling of a $2^n \times 2^n$ courtyard with Bill in a central square.*

[2] In the special case $n = 0$, the whole courtyard consists of a single central square; otherwise, there are four central squares.