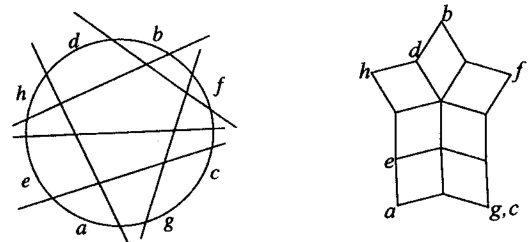
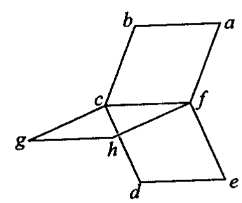
$\{a,b,d,e,h\} \mid \{c,f,g\}$
$\{a,c,d,e,g,h\} \mid \{b,f\}$
$\{a,c,e,g\} \mid \{b,d,f,h\}$
$\{a,c,g\} \mid \{b,d,e,f,h\}$
$\{a,c,e,f,g\} \mid \{b,d,h\}$
$\{a,e,h\} \mid \{b,c,d,f,g\}$



(a) Circular splits (b) Circular ordering (c) Planar network

Figure 5.9 (a) A set of six circular splits $\mathcal{S}$ on $\mathcal{X} = \{a, b, \ldots, h\}$. (b) An arrangement of the taxa around a circle such that every split $S = A \mid B \in \mathcal{S}$ can be realized by a straight line through the circle that separates the two split parts A and B. A circular ordering is given by (a, g, c, f, b, d, h, e). (c) An outer-labeled planar split network representing $\mathcal{S}$.

$\{a,b\} \mid \{c,d,e,f,g,h\}$
$\{a,b,c,d,e,f\} \mid \{g,h\}$
$\{a,b,c,f,g,h\} \mid \{d,e\}$
$\{a,e,f,h\} \mid \{b,c,d,g\}$



(a) Non-circular splits (b) Non-planar network

Figure 5.10 (a) A set of four non-circular splits $\mathcal{S}$ on $\mathcal{X} = \{a, b, \ldots, h\}$. (b) A non-planar split network representing $\mathcal{S}$.

5.7 Circular splits and planar split networks

One practical problem that arises when working with split networks is that the networks can be very complicated and thus difficult to visualize in a comprehensible way. Hence, a number of restricted classes of sets of splits have been introduced in an attempt to avoid overly complicated networks. The two most important are *circular* splits, which are the focus of this section, and *weakly compatible* splits, which we introduce in the next section.

Informally, a set of splits $\mathcal{S}$ on $\mathcal{X}$ is called *circular*, if the taxa in $\mathcal{X}$ can be placed around a circle in such a way that each split $S = \frac{A}{B}$ can be realized by a line through the circle that separates the plane into two half-planes, one containing all taxa in A and the other containing all taxa in B (see Figure 5.9). An example of a non-circular set of splits and the corresponding split network is shown in Figure 5.10. More formally [9]:

Definition 5.7.1 (Circular splits) *A set of splits $\mathcal{S}$ on $\mathcal{X}$ is called* circular, *if there exists a linear ordering $(x_1, \ldots, x_n)$ of the elements of $\mathcal{X}$* for *$\mathcal{S}$ such that each split*

$$\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array}$$
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(a) Input matrix

$$\begin{array}{cccccccc} a & e & h & d & b & f & c & g \end{array}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(b) Permuted matrix

Figure 5.11 (a) An input matrix M for the Consecutive Ones problem that corresponds to the set of splits shown in Figure 5.9(a). (b) A solution obtained by permuting the columns of M in the order suggested by Figure 5.9(b).

$S \in \mathcal{S}$ has the form

$$S = \frac{\{x_p, x_{p+1}, \ldots, x_q\}}{\mathcal{X} - \{x_p, x_{p+1}, \ldots, x_q\}}, \tag{5.17}$$

for appropriately chosen $1 < p \leq q \leq n$.

We call such an ordering $(x_1, \ldots, x_n)$ a *circular ordering* for $\mathcal{S}$, as it holds that $(x_1, \ldots, x_n)$ is a circular ordering for $\mathcal{S}$ if and only if $(x_n, x_{n-1}, \ldots, x_1)$ and $(x_2, x_3, \ldots, x_n, x_1)$ both are. How to determine whether a set of splits $\mathcal{S}$ is circular? This is equivalent to a well-known problem in computer science:

Problem 5.7.2 (Consecutive Ones problem) *Let M be a binary matrix. Does there exist a permutation of the columns of the matrix M such that in each row, all ones in the row occur in a single consecutive run?*

It is straightforward to translate the problem of determining whether $\mathcal{S}$ is circular into an instance of the Consecutive Ones problem. Simply define a binary matrix M in which each row r corresponds to some split $S \in \mathcal{S}$ and every column c corresponds to some taxon $x \in \mathcal{X}$. Then set $M(r, c) = 1$ if and only if S separates x from the taxon represented by the first column of M.

Exercise 5.7.3 (Circular ordering and Consecutive Ones) *Prove the following statement: There exists a circular ordering of $\mathcal{X}$ for $\mathcal{S}$ if and only if a solution of the Consecutive Ones problem exists for M. Hint: construct a binary matrix M whose rows represent splits and whose columns represent taxa (see Figure 5.11).*

The Consecutive Ones problem can be solved in linear time [23]. However, if no circular ordering exists for a given set of splits $\mathcal{S}$ on $\mathcal{X}$, then one might try to determine a solution that is as good as possible. For our purposes, an appropriate optimization goal is to find an ordering of $\mathcal{X}$ that minimizes the number of runs