is constructed when $F < M - |V|^{1/2}$. This layered network is identical with the first for $\tilde{G}$, with zero flow everywhere. Also, by Lemma 6.5, $G$ is of type 2. Thus, by Lemma 6.4, the length $l$ of the layered network is at most $(|V| - 2) / \tilde{M} + 1$. Now, $\tilde{M} = M - F > M - (M - |V|^{1/2}) = |V|^{1/2}$.

Thus,

$$l \leq \frac{|V| - 2}{|V|^{1/2}} + 1 = O(|V|^{1/2}).$$

Therefore, the number of phases up to this one is at most $O(|V|^{1/2})$. Since the number of phases to completion is at most $|V|^{1/2}$ more, the total number of phases is at most $O(|V|^{1/2})$.

Q.E.D.

6.2 VERTEX CONNECTIVITY OF GRAPHS

Intuitively, the connectivity of a graph is the minimum number of elements whose removal from the graph disconnect it to more than one component. There are four cases. We may discuss undirected graphs or digraphs; we may discuss the elimination of edges or vertices. We shall start with the problem of determining the vertex-connectivity of an undirected graph. The other cases, which are simpler, will be discussed in the next section.

Let $G(V, E)$ be a finite undirected graph, with no self-loops and no parallel edges. A set of vertices, $S$, is called an $(a, b)$ vertex separator if $\{a, b\} \subset V - S$ and every path connecting $a$ and $b$ passes through at least one vertex of $S$. Clearly, if $a$ and $b$ are connected by an edge, no $(a, b)$ vertex separator exists. Let $a \not\sim b$ mean that there is no such edge. In this case, let $N(a, b)$ be the least cardinality of an $(a, b)$ vertex separator. Also, let $p(a, b)$ be the maximum number of pairwise vertex disjoint paths connecting $a$ and $b$ in $G$; clearly, all these paths share the two end-vertices, but no other vertex appears on more than one of them.

Theorem 6.4: If $a \not\sim b$ then $N(a, b) = p(a, b)$.

This is one of the variations of Menger's theorem [2]. It is not only reminiscent of the max-cut min-flow theorem, but can be proved by it. Dantzig and Fulkerson [3] pointed out how this can be done, and we shall follow their approach.

Proof: Construct a digraph $\tilde{G}(\tilde{V}, \tilde{E})$ as follows. For every $v \in V$ put two vertices $v'$ and $v''$ in $\tilde{V}$ with an edge $v' \not\sim v''$. For every edge $u \sim v$ in $G$, put
two edges $u'' \rightarrow v'$ and $v'' \rightarrow u'$ in $\bar{G}$. Define now a network, with digraph $\bar{G}$, source $a''$, sink $b'$, unit capacities for all the edges of the $e'$ type (let us call them internal edges), and infinite capacity for all the edges of the $e'$ and $e''$ type (called external edges). For example, in Fig. 6.1(b) the network for $G$, as shown in Fig. 6.1(a), is demonstrated.

We now claim that $p(a, b)$ is equal to the total maximum flow $F$ (from $a''$ to $b'$) in the corresponding network. First, assume we have $p(a, b)$ vertex

![Diagram](a)

![Diagram](b)

**Figure 6.1**