Solution to problem 3 of HW 4.

Theorem: Assume that the connectivity of \( G \) is less than the smallest node degree in \( G \), and let \( u \) be an arbitrary node in \( G \). Then the connectivity of \( G \) is equal to \( \min[c(u, v) : v \neq u \text{ has the same parity as } u] \).

Hence to compute the connectivity, we choose an odd node \( u \) if the number of odd nodes is less than the number of even nodes, else we choose an even node \( u \). Then for each node \( v \) with the same parity as \( u \), we compute the minimum cut between \( u \) and \( v \), and take the minimum of those cut values. The edge connectivity of \( G \) is then the smaller of the minimum node degree, and the minimum of those cut values.

Your problem 3: Prove the Theorem.

(This might be a hard problem)

Answer: The first answer is the one I had prepared before making the assignment. But some students actually found nicer proofs, so I include one as well.

\textbf{Proof 1:} Let \( K \) denote the connectivity of \( G \). First, we claim that \( |X| > K \) and \( |Y| > K \). Let \( \delta \) denote the smallest node degree in \( G \), and \( e(X) \) be the number of edges inside \( X \), i.e., with both endpoints in \( X \). The sum of all the degrees of the nodes in \( X \) is \( 2e(X) + K \) which is at least \( \delta |X| \) so \( 2e(X) + K \geq \delta |X| \). Now \( e(X) \leq |X|(|X| - 1)/2 \), so \( |X|(|X| - 1) + K \geq \delta |X| \).

Now \( K < \delta \), so \( K|X| < |X|(|X| - 1) + K \), so \( K(|X| - 1) < |X|(|X| - 1) \).

If \( |X| > 1 \), then we can divide by \( |X| - 1 \) to get \( K < |X| \) as claimed. But certainly \( |X| > 1 \), since \( K < \delta \), so \( |X| > K \). The same argument shows \( |Y| > K \).

Second, since \( |X| > K \), which is the number of edges that cross between node sets \( X \) and \( Y \), at least one node in \( X \) is not adjacent to any node in \( Y \), and similarly some node in \( Y \) is not adjacent to any node in \( X \).

Now we define a dominating set of nodes, \( D \), as a set such that every node in \( G \) is either in \( D \) or is adjacent to a node in \( D \). Since there is a node in \( X \) that is not adjacent to any node in \( Y \), and a node in \( Y \) that is not adjacent to any node in \( X \), any dominating set in \( G \) must contain at least one node from \( X \) and at least one node from \( Y \). So if we pick any node \( u \) in a dominating set, and then successively compute the minimum cut capacity between \( u \) and each other node in the dominating set, the minimum of those capacities will be \( K \), assuming \( K < \delta \).

Finally, note that the set of odd nodes in the spanning tree is a dominating set of \( G \), as is the set of even nodes. That is, \( K = \min[c(u, v) : v \neq u] \).
u has the same parity as u]. Now one of those sets has size less than or equal to n/2, so n/2 min-cut computations suffice to find K.

Proof 2:

For concreteness, and WLOG, assume that node u (from the algorithm) is in X (it has to either be in X or Y), and is in the set of nodes of even parity.

Suppose that the Theorem is not correct. Then Y can only contain nodes of odd parity, for it Y contained a node v of even parity, the algorithm would compute the minimum u, v cut and so would compute the capacity of the cut (X, Y). In the spanning tree, every node of odd parity is adjacent to some node of even parity, so every node in Y must be adjacent to some node in X. Let v be any node in Y, and let x_v be the number of nodes in X that v is adjacent to. So, the capacity of the cut (X, Y) must be at least \( x_v + |Y| - 1 \).

On the other hand, the degree of v is at most \( x_v + |Y| - 1 \), contradicting the assumption that the capacity of (X, Y) is less than the minimum degree of any node in G.