Exercise 1

a) Show that $2x - 10$ is $\Theta(x)$.

One option is to prove that $2x - 10$ is both $O(x)$ and $\Omega(x)$. In this simple case however, we directly “squeeze” $2x - 10$ between two functions that are of order $x$. First, let us notice that $\forall x \in \mathbb{R}, 2x - 10 < 2x$.

Second, we note that if $x > 10$, then $x - 10 > 0$ and therefore $2x - 10 > x$

Summarizing: for $x > 10$, $x < 2x - 10 < 2x$. Therefore $2x - 10$ is $\Theta(x)$.

b) Show that $4x^2 + 8x - 6$ is $\Theta(x^2)$.

Again, one option is to prove that $4x^2 + 8x - 6$ is both $O(x^2)$ and $\Omega(x^2)$. In this simple case however, we directly “squeeze” $4x^2 + 8x - 6$ between two functions that are of order $x^2$.

We note first that when $8x > 6$, then $8x - 6 > 0$, and therefore $4x^2 + 8x - 6 > 4x^2$.

Second, we note that when $1 < x$, $x < x^2$, and therefore $8x < 8x^2$. We also have $-6 < x^2$. This leads to $4x^2 + 8x - 6 < 13x^2$ when $x > 1$.

Summarizing: for $x > 1$, $4x^2 < 4x^2 + 8x - 6 < 13x^2$. Therefore $4x^2 + 8x - 6$ is $\Theta(x^2)$.

c) Show that $\lfloor x + \frac{2}{7} \rfloor$ is $\Theta(x)$.

Again, we will “squeeze” $\lfloor x + \frac{2}{7} \rfloor$ between two functions that are of order $x$.

By definition of the function floor, $\lfloor x + \frac{2}{7} \rfloor \leq x + \frac{2}{7}$. If $\frac{2}{7} < x$, this leads to $\lfloor x + \frac{2}{7} \rfloor < 2x$.

Similarly, $x + \frac{2}{7} < \lfloor x + \frac{2}{7} \rfloor + 1$, therefore $x - \frac{5}{7} < \lfloor x + \frac{2}{7} \rfloor$.

If $x > 1$, then $-x < -1$; multiplying by $\frac{5}{7}$, $-\frac{5x}{7} < -\frac{5}{7}$, and adding $x$, we get $\frac{2x}{7} < x - \frac{5}{7}$, therefore $\frac{2x}{7} < \lfloor x + \frac{2}{7} \rfloor$.

Summarizing: for $x > 1$, $\frac{2x}{7} < \lfloor x + \frac{2}{7} \rfloor < 2x$. Therefore $\lfloor x + \frac{2}{7} \rfloor$ is $\Theta(x)$.

d) Show that $\log_4(x)$ is $\Theta \log_7(x)$.

Notice first that $\log_4(x) = \log_7(4) \times \log_7(x)$. Since $\log_4(x)$ and $\log_7(x)$ only differ by a (positive) constant, there are of the same order. Hence $\log_4(x)$ is $\Theta(\log_7(x))$. 

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Homework 6 Solutions

ECS 20 (Winter 2019)

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Exercise 2

Show that \( x^2 \) is \( O(x^4) \) but that \( x^4 \) is not \( O(x^3) \).

a) Let us show that \( x^2 \) is \( O(x^4) \).

Let us assume that \( 1 < x \). Since \( x > 0 \), we can multiply this inequality by \( x \): \( x^2 < x^3 \), and finally \( x^3 < x^4 \). As \( x^2 < x^3 \) and \( x^3 < x^4 \), we have \( x^2 < x^4 \).

We have shown that there exists \( k \) (\( k = 1 \)), and there exists \( C \) (\( C = 1 \)), such that if \( x > k \), then \( x^2 < Cx^4 \); we can conclude that \( x^2 \) is \( O(x^4) \).

b) Let us show that \( x^4 \) is not \( O(x^2) \).

We use a proof by contradiction: let us suppose that the proposition is true, i.e. that \( x^4 \) is \( O(x^2) \). By definition of \( O \), this means that:
\[ \exists k \in \mathbb{R}, \exists C \in \mathbb{R} \text{ if } x > k \text{ then } |x^4| < C|x^2|. \]

Let \( D = \max\{2, k, C\} \). Therefore \( D > 1 \), \( D \geq k \), and \( D \geq C \).

Let \( x \) be a real number with \( x > D \). Since \( D > 1 \), \( x > 1 \) and therefore \( x^2 > x > D \).

Since \( D \geq k \), we have \( x^4 < Cx^2 < Dx^2 \). Since \( x > 0 \), we can divide this inequality by \( x^2 \): we get \( x^2 < D \). We have shown that \( x^2 > D \) and \( x^2 < D \); we have reached a contradiction.

Therefore, the hypothesis, \( x^4 \) is \( O(x^2) \), is false. We can conclude that \( x^4 \) is not \( O(x^2) \).

Exercise 3

Let \( a \), and \( b \) be two strictly positive integers and let \( x \) be a real number. Show that:

\[
\left\lfloor \left\lfloor \frac{x}{a} \right\rfloor \frac{1}{b} \right\rfloor = \left\lfloor \frac{x}{ab} \right\rfloor
\]

Let us define \( k = \left\lfloor \frac{x}{a} \right\rfloor \) and \( m = \left\lfloor \frac{x}{ab} \right\rfloor \). By definition of floor, we have the two properties:
\( k \leq \frac{x}{a} < k + 1 \)
and
\( m \leq \frac{x}{ab} < m + 1 \)

Let us multiply the second inequalities by \( b \):
\( bm \leq \frac{x}{a} < b(m + 1) \)

We notice that:
\( k \leq \frac{x}{a} \) and \( \frac{x}{a} < b(m + 1) \); therefore \( k < b(m + 1) \).
\( k \leq \frac{x}{a} \) and \( bm \leq \frac{x}{a} \). Therefore \( k \) and \( bm \) are two integers smaller than \( \frac{x}{a} \). By definition of floor, \( k \) is the largest integer smaller that \( \frac{x}{a} \). Therefore \( bm \leq k \).

Combining those two inequalities, we get \( bm \leq k < b(m + 1) \). After division by \( b \), \( m < \frac{k}{b} < m + 1 \).
Therefore \( m \) is the floor of \( \frac{k}{b} \). Replacing \( m \) and \( k \) by their values, we get:

\[
m = \left\lfloor \frac{x}{ab} \right\rfloor = \left\lfloor \frac{k}{b} \right\rfloor = \left\lfloor \frac{x}{a} \right\rfloor \frac{1}{b}
\]

The property is therefore true.
Exercise 4

Let $x$ be a positive real number. Solve $\lfloor x\lfloor x\rfloor \rfloor = 5$.

Let $A = \lfloor x\lfloor x\rfloor \rfloor$.
Since $x \geq 0$, we do not need to worry about $x$ being negative.
We notice first that if $x \geq 3$, then $\lfloor x\rfloor \geq 3$, and $x\lfloor x\rfloor \geq 9$, therefore $A \geq 9$.
Therefore possible solutions for $x$ are between 0 and 3, 3 not included. We look at three cases:

a) $0 \leq x < 1$
In this case, $\lfloor x\rfloor = 0$ and $A = 0$. There are no solutions in this interval.

b) $1 \leq x < 2$
In this case, $\lfloor x\rfloor = 1$ and $A = \lfloor x\rfloor = 1$. There are no solutions in this interval.

c) $2 \leq x < 3$
In this case, $\lfloor x\rfloor = 2$ and $A = \lfloor 2x\rfloor$. Since $2 \leq x < 3$, $4 \leq 2x < 6$. We distinguish two cases:

i) $4 \leq 2x < 5$, namely $2 \leq x < 2.5$. Then $A = \lfloor 2x\rfloor = 4$; there are no solutions in this interval.

ii) $5 \leq 2x < 6$, namely $2.5 \leq x < 3$. Then $A = \lfloor 2x\rfloor = 5$; all values of $x$ in this interval are solutions.

In conclusion, all values of $x \in [2.5, 3]$ are solutions of the equation.

Exercise 5

Let $n$ be a natural number. Show that if $n$ is a perfect square, then $2n$ is not a perfect square.

We will do a proof by contradiction. The property is of the form $P : p \rightarrow q$, where $p$ is “$n$ is a perfect square” and $q$ is “$2n$ is not a perfect square”. Assuming $P$ is false is equivalent to assuming that $p$ is true AND $q$ is false. Therefore:

$p$ is true: there exists an integer $k$ such that $n = k^2$
$q$ is false: there exists an integer $l$ such that $2n = l^2$.

Since $n > 0$, $k \neq 0$ and $l \neq 0$. Replacing $n$ with $k^2$ in the second equality, we get,

$2k^2 = l^2$. Taking the square root, we get $\sqrt{2}k = l$... but this would say that $\sqrt{2}$ is rational.
We have reached a contradiction: the property $\neg P$ is therefore false, and then $P$ is true.

Extra Credit

Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy:

$\forall (x, y) \in \mathbb{R}^2, f(x)f(y) + f(x + y) = xy$

As the property is true for all pairs of real number, it is true for $(x, y) = (0, 0)$. Therefore:

$f(0)^2 + f(0) = 0$

from which we deduce that $f(0) = 0$ or $f(0) = -1$. 
a) \( f(0) = 0 \)

Applying the property to \((x, y) = (a, 0)\), where \(a\) is a real number, we get \(f(a) = 0\) for all \(a\), therefore \(f\) is the null function.

b) \( f(0) = -1 \)

Applying the property to \((x, y) = (1, -1)\), we get: \(f(1)f(-1) + f(0) = -1\), i.e. \(f(1)f(-1) = 0\), i.e. \(f(1) = 0\) or \(f(-1) = 0\).

i) \( f(1) = 0 \).

We apply the property to \((x, y) = (a-1, 1)\), we get: \(f(a-1)f(1) + f(a) = a - 1\), therefore \(f(a) = a - 1\)

ii) \( f(-1) = 0 \).

We apply the property to \((x, y) = (a + 1, -1)\), we get: \(f(a + 1)f(-1) + f(a) = -a - 1\), therefore \(f(a) = -a - 1\)

We have found that if \(f\) satisfies the property, then \(f\) is one of the three following functions: 
\(f_1(x) = 0\), \(f_2(x) = x - 1\) and \(f_3(x) = -x - 1\). We note however that \(f_1(x)\) does not satisfy the property: let \((x, y)\) be two real numbers; \(f(x)f(y) + f(x + y) = 0\), while \(xy = 0\) if and only if \(x = 0\) or \(y = 0\). For the other two functions,
\(f_2(x)f_2(y) + f_2(x + y) = (x - 1)(y - 1) + x + y - 1 = xy\)
and
\(f_3(x)f_3(y) + f_3(x + y) = (x + 1)(y + 1) - x - y - 1 = xy\)

Therefore \(f_2\) and \(f_3\) satisfy the property. They are the only solutions.