4.19 Consider the problem

\[
\begin{align*}
& \text{minimize} & & \|Ax - b\|_1 / (c^T x + d) \\
& \text{subject to} & & \|x\|_\infty \leq 1,
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), \( c \in \mathbb{R}^n \), and \( d \in \mathbb{R} \). We assume that \( d > \|c\|_1 \), which implies that \( c^T x + d > 0 \) for all feasible \( x \).

(a) Show that this is a quasiconvex optimization problem.
(b) Show that it is equivalent to the convex optimization problem

\[
\begin{align*}
& \text{minimize} & & \|Ay - bt\|_1 \\
& \text{subject to} & & \|y\|_\infty \leq t \\
& & & c^T y + dt = 1,
\end{align*}
\]

with variables \( y \in \mathbb{R}^n \), \( t \in \mathbb{R} \).

Solution.

(a) \( f_0(x) \leq \alpha \) if and only if

\[
\|Ax - b\|_1 - \alpha (c^T x + d) \leq 0,
\]

which is a convex constraint.

(b) Suppose \( \|x\|_\infty \leq 1 \). We have \( c^T x + d > 0 \), because \( d > \|c\|_1 \). Define

\[
y = x / (c^T x + d), \quad t = 1 / (c^T x + d).
\]

Then \( y \) and \( t \) are feasible in the convex problem with objective value

\[
\|Ay - bt\|_1 = \|Ax - b\|_1 / (c^T x + d).
\]

Conversely, suppose \( y, t \) are feasible for the convex problem. We must have \( t > 0 \), since \( t = 0 \) would imply \( y = 0 \), which contradicts \( c^T y + dt = 1 \). Define

\[
x = y / t.
\]

Then \( \|x\|_\infty \leq 1 \), and \( c^T x + d = 1 / t \), and hence

\[
\|Ax - b\|_1 / (c^T x + d) = \|Ay - bt\|_1.
\]

4.25 Linear separation of two sets of ellipsoids. Suppose we are given \( K + L \) ellipsoids

\[
\mathcal{E}_i = \{ Pu + q_i \mid \|u\|_2 \leq 1 \}, \quad i = 1, \ldots, K + L,
\]

where \( P_i \in \mathbb{S}^n \). We are interested in finding a hyperplane that strictly separates \( \mathcal{E}_1, \ldots, \]

\( \mathcal{E}_K \) from \( \mathcal{E}_{K+1}, \ldots, \mathcal{E}_{K+L} \), i.e., we want to compute \( a \in \mathbb{R}^n \), \( b \in \mathbb{R} \) such that

\[
a^T x + b > 0 \quad \text{for} \; x \in \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_K, \quad a^T x + b < 0 \quad \text{for} \; x \in \mathcal{E}_{K+1} \cup \cdots \cup \mathcal{E}_{K+L},
\]

or prove that no such hyperplane exists. Express this problem as an SOCP feasibility problem.

Solution. We first note that the problem is homogeneous in \( a \) and \( b \), so we can replace the strict inequalities \( a^T x + b > 0 \) and \( a^T x + b < 0 \) with \( a^T x + b \geq 1 \) and \( a^T x + b \leq -1 \), respectively.

\( a \) and \( b \) must satisfy

\[
\inf_{i=1}^{L} (a^T P_i a + a^T q_i) = -\|P_i^T a\|_2 + a^T q_i + b \geq 1, \quad i = 1, \ldots, L
\]

\[
\sup_{i=K+1}^{K+L} (a^T P_i a + a^T q_i) = \|P_i^T a\|_2 + a^T q_i + b \leq -1, \quad i = K + 1, \ldots, K + L.
\]

These form a set of SOC constraints in \( a, b \).
Solution.

(a) The objective function is a maximum of convex function, hence convex.
We can write the problem as

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad (1/2)x^TPx + q^Tx + r \leq t, \quad i = 1, \ldots, K \\
& \quad Ax \leq b_i
\end{align*}
\]

which is a QCQP in the variable \(x\) and \(t\).

4.30 A heated fluid at temperature \(T\) (degrees above ambient temperature) flows in a pipe with fixed length and circular cross section with radius \(r\). A layer of insulation, with thickness \(w \ll r\), surrounds the pipe to reduce heat loss through the pipe walls. The design variables in this problem are \(T\), \(r\), and \(w\).

The heat loss is (approximately) proportional to \(Tr/w\), so over a fixed lifetime, the energy cost due to heat loss is given by \(\alpha_1 Tr/w\). The cost of the pipe, which has a fixed wall thickness, is approximately proportional to the total material, i.e., it is given by \(\alpha_2 r\). The cost of the insulation is also approximately proportional to the total insulation material, i.e., \(\alpha_3 rw\) (using \(w \ll r\)). The total cost is the sum of these three costs.

The heat flow down the pipe is entirely due to the flow of the fluid, which has a fixed velocity, i.e., it is given by \(\alpha_4 Tr^2\). The constants \(\alpha_i\) are all positive, as are the variables \(T\), \(r\), and \(w\).

Now the problem: maximize the total heat flow down the pipe, subject to an upper limit \(C_{\text{max}}\) on total cost, and the constraints

\[
T_{\text{min}} \leq T \leq T_{\text{max}}, \quad r_{\text{min}} \leq r \leq r_{\text{max}}, \quad w_{\text{min}} \leq w \leq w_{\text{max}}, \quad w \leq 0.1r.
\]

Express this problem as a geometric program.

Solution. The problem is

\[
\begin{align*}
\text{maximize} & \quad \alpha_4 Tr^2 \\
\text{subject to} & \quad \alpha_1 T w^{-1} + \alpha_2 r + \alpha_3 rw \leq C_{\text{max}} \\
& \quad T_{\text{min}} \leq T \leq T_{\text{max}} \\
& \quad r_{\text{min}} \leq r \leq r_{\text{max}} \\
& \quad w_{\text{min}} \leq w \leq w_{\text{max}} \\
& \quad w \leq 0.1r.
\end{align*}
\]

This is equivalent to the GP

\[
\begin{align*}
\text{minimize} & \quad (1/\alpha_4)T^{-1}r^{-2} \\
\text{subject to} & \quad (\alpha_1/C_{\text{max}})T w^{-1} + (\alpha_2/C_{\text{max}})r + (\alpha_3/C_{\text{max}})rw \leq 1 \\
& \quad (1/T_{\text{max}})T \leq 1, \quad T_{\text{min}} T^{-1} \leq 1 \\
& \quad (1/r_{\text{max}})r \leq 1, \quad r_{\text{min}} r^{-1} \leq 1 \\
& \quad (1/w_{\text{max}})w \leq 1, \quad w_{\text{min}} w^{-1} \leq 1 \\
& \quad 10wr^{-1} \leq 1.
\end{align*}
\]
4.43 **Eigenvalue optimization via SDP.** Suppose \( A : \mathbb{R}^n \to \mathbb{S}^m \) is affine, i.e.,
\[
A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n
\]
where \( A_i \in \mathbb{S}^m \). Let \( \lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x) \) denote the eigenvalues of \( A(x) \). Show how to pose the following problems as SDPs.

(a) Minimize the maximum eigenvalue \( \lambda_1(x) \).

(b) Minimize the spread of the eigenvalues, \( \lambda_1(x) - \lambda_m(x) \).

(c) Minimize the condition number of \( A(x) \), subject to \( A(x) \succeq 0 \). The condition number is defined as \( \kappa(A(x)) = \lambda_1(x)/\lambda_m(x) \), with domain \( \{ x \mid A(x) \succeq 0 \} \). You may assume that \( A(x) \succeq 0 \) for at least one \( x \).

*Hint.* You need to minimize \( \gamma I \), subject to
\[
0 < \gamma I \preceq A(x) \preceq \lambda I.
\]
Change variables to \( y = x/\gamma \), \( t = \lambda/\gamma \), \( s = 1/\gamma \).

(d) Minimize the sum of the absolute values of the eigenvalues, \( |\lambda_1(x)| + \cdots + |\lambda_m(x)| \).

*Hint.* Express \( A(x) \) as \( A(x) = A_+ - A_- \), where \( A_+ \succeq 0 \), \( A_- \succeq 0 \).

**Solution.**

(a) We use the property that \( \lambda_1(x) \leq t \) if and only if \( A(x) \preceq tI \). We minimize the maximum eigenvalue by solving the SDP
\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad A(x) \preceq tI.
\end{align*}
\]
The variables are \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R} \).

(b) \( \lambda_1(x) \leq t_1 \) if and only if \( A(x) \preceq t_1 I \) and \( \lambda_m(A(x)) \geq t_2 \) if and only if \( A(x) \succeq t_2 I \), so we can minimize \( \lambda_1 - \lambda_m \) by solving
\[
\begin{align*}
\text{minimize} & \quad t_1 - t_2 \\
\text{subject to} & \quad t_2 I \preceq A(x) \preceq t_1 I.
\end{align*}
\]
This is an SDP with variables \( t_1 \in \mathbb{R} \), \( t_2 \in \mathbb{R} \), and \( x \in \mathbb{R}^n \).

(c) We first note that the problem is equivalent to
\[
\begin{align*}
\text{minimize} & \quad \lambda/\gamma \\
\text{subject to} & \quad \gamma I \preceq A(x) \preceq \lambda I
d\end{align*}
\]
if we take as domain of the objective \( \{(\lambda, \gamma) \mid \gamma > 0\} \). This problem is quasiconvex, and can be solved by bisection: The optimal value is less than or equal to \( \alpha \) if and only if the inequalities
\[
\lambda \leq \gamma \alpha, \quad \gamma I \preceq A(x) \preceq \lambda I, \quad \gamma > 0
\]
(with variables \( \gamma, \lambda, x \) are feasible).

Following the hint we can also pose the problem as the SDP
\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad I \preceq s A_0 + y_1 A_1 + \cdots + y_n A_n \preceq tI \\
& \quad s \geq 0.
\end{align*}
\]
We now verify more carefully that the two problems are equivalent. Let \( p^* \) be the optimal value of (4.43.A), and \( f_{\text{sdp}}^* \) is the optimal value of the SDP (4.43.B).
5.12 Analytic centering. Derive a dual problem for

$$\text{minimize} \quad -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

with domain \( \{ x \mid a_i^T x < b_i, \ i = 1, \ldots, m \} \). First introduce new variables \( y_i \) and equality constraints \( y_i = b_i - a_i^T x \).

(The solution of this problem is called the analytic center of the linear inequalities \( a_i^T x \leq b_i, \ i = 1, \ldots, m \). Analytic centers have geometric applications (see §8.5.3), and play an important role in barrier methods (see chapter 11).)

Solution. We derive the dual of the problem

$$\text{minimize} \quad -\sum_{i=1}^{m} \log y_i$$

subject to \( y = b - Ax \),

where \( A \in \mathbb{R}^{n \times n} \) has \( a_i^T \) as its \( i \)th row. The Lagrangian is

$$L(x, y, \nu) = -\sum_{i=1}^{m} \log y_i + \nu^T (y - b + Ax)$$

and the dual function is

$$g(\nu) = \inf_{x, y} \left( -\sum_{i=1}^{m} \log y_i + \nu^T (y - b + Ax) \right).$$

The term \( \nu^T Ax \) is unbounded below as a function of \( x \) unless \( A^T \nu = 0 \). The terms in \( y \) are unbounded below if \( \nu \neq 0 \), and achieve their minimum for \( y_i = 1/\nu_i \), otherwise. We therefore find the dual function

$$g(\nu) = \begin{cases} 
\sum_{i=1}^{m} \log \nu_i + m - b^T \nu & A^T \nu = 0, \ \nu > 0 \\
-\infty & \text{otherwise}
\end{cases}$$

and the dual problem

$$\text{maximize} \quad \sum_{i=1}^{m} \log \nu_i - b^T \nu + m$$

subject to \( A^T \nu = 0 \).
5.13 Lagrangian relaxation of Boolean LP. A Boolean linear program is an optimization problem of the form

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x_i \in \{0, 1\}, \quad i = 1, \ldots, n,
\end{align*}
\]

and is, in general, very difficult to solve. In exercise 4.15 we studied the LP relaxation of this problem,

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad 0 \leq x_i \leq 1, \quad i = 1, \ldots, n,
\end{align*}
\]

which is far easier to solve, and gives a lower bound on the optimal value of the Boolean LP. In this problem we derive another lower bound for the Boolean LP, and work out the relation between the two lower bounds.

(a) Lagrangian relaxation. The Boolean LP can be reformulated as the problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x_i(1 - x_i) = 0, \quad i = 1, \ldots, n,
\end{align*}
\]

which has quadratic equality constraints. Find the Lagrange dual of this problem. The optimal value of the dual problem (which is convex) gives a lower bound on the optimal value of the Boolean LP. This method of finding a lower bound on the optimal value is called Lagrangian relaxation.

(b) Show that the lower bound obtained via Lagrangian relaxation, and via the LP relaxation (5.107), are the same. \textit{Hint.} Derive the dual of the LP relaxation (5.107).

Solution.

(a) The Lagrangian is

\[
L(x, \mu, \nu) = c^T x + \mu^T (Ax - b) - \nu^T x + x^T \text{diag}(\nu) x
\]

\[
= x^T \text{diag}(\nu) x + (c + A\mu - \nu)^T x - b^T \mu.
\]

Minimizing over \( x \) gives the dual function

\[
g(\mu, \nu) = \begin{cases} 
-b^T \mu - (1/4) \sum_{i=1}^n (c_i + a_i^T \mu - \nu) / \nu_i & \nu \geq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

where \( a_i \) is the \( i \)-th column of \( A \), and we adopt the convention that \( a_i^2/0 = \infty \) if \( a \neq 0 \), and \( a_i^2/0 = 0 \) if \( a = 0 \).

The resulting dual problem is

\[
\begin{align*}
\text{maximize} & \quad -b^T \mu - (1/4) \sum_{i=1}^n (c_i + a_i^T \mu - \nu_i) / \nu_i \\
\text{subject to} & \quad \nu \geq 0.
\end{align*}
\]

In order to simplify this dual, we optimize analytically over \( \nu \), by noting that

\[
\sup_{\nu_i \geq 0} \left( \frac{(c_i + a_i^T \mu - \nu_i)}{\nu_i} \right) = \begin{cases} 
(c_i + a_i^T \mu) & c_i + a_i^T \mu \leq 0 \\
0 & c_i + a_i^T \mu > 0
\end{cases}
\]

\[
= \min(0, (c_i + a_i^T \mu)).
\]

This allows us to eliminate \( \nu \) from the dual problem, and simplify it as

\[
\begin{align*}
\text{maximize} & \quad -b^T \mu + \sum_{i=1}^n \min(0, c_i + a_i^T \mu) \\
\text{subject to} & \quad \mu \geq 0.
\end{align*}
\]
(b) We follow the hint. The Lagrangian and dual function of the LP relaxation re

\[ L(x, u, v, w) = c^T x + u^T (Ax - b) - v^T x + w^T (x - 1) \]

\[ = (c + A^T u - v + w)^T x - b^T u - 1^T w \]

\[ g(u, v, w) = \begin{cases} 
-b^T u - 1^T w & A^T u - v + w + c = 0 \\
-\infty & \text{otherwise.} 
\end{cases} \]

The dual problem is

maximize \(-b^T u - 1^T w\)

subject to \(A^T u - v + w + c = 0\)
\(u \geq 0, v \geq 0, w \geq 0,\)

which is equivalent to the Lagrange relaxation problem derived above. We conclude that the two relaxations give the same value.
4.30 A heated fluid at temperature $T$ (degrees above ambient temperature) flows in a pipe with fixed length and circular cross section with radius $r$. A layer of insulation, with thickness $w \ll r$, surrounds the pipe to reduce heat loss through the pipe walls. The design variables in this problem are $T$, $r$, and $w$.

The heat loss is (approximately) proportional to $Tr/w$, so over a fixed lifetime, the energy cost due to heat loss is given by $\alpha_1 Tr/w$. The cost of the pipe, which has a fixed wall thickness, is approximately proportional to the total material, i.e., it is given by $\alpha_2 r$. The cost of the insulation is also approximately proportional to the total insulation material, i.e., $\alpha_3 rw$ (using $w \ll r$). The total cost is the sum of these three costs.

The heat flow down the pipe is entirely due to the flow of the fluid, which has a fixed velocity, i.e., it is given by $\alpha_4 Tr^2$. The constants $\alpha_i$ are all positive, as are the variables $T$, $r$, and $w$.

Now the problem: maximize the total heat flow down the pipe, subject to an upper limit $C_{\text{max}}$ on total cost, and the constraints

$$T_{\text{min}} \leq T \leq T_{\text{max}}, \quad r_{\text{min}} \leq r \leq r_{\text{max}}, \quad w_{\text{min}} \leq w \leq w_{\text{max}}, \quad w \leq 0.1r.$$ 

Express this problem as a geometric program.

**Solution.** The problem is

$$\begin{align*}
\text{maximize} & \quad \alpha_4 Tr^2 \\
\text{subject to} & \quad \alpha_1 Tw^{-1} + \alpha_2 r + \alpha_3 rw \leq C_{\text{max}} \\
& \quad T_{\text{min}} \leq T \leq T_{\text{max}} \\
& \quad r_{\text{min}} \leq r \leq r_{\text{max}} \\
& \quad \alpha_1/C_{\text{max}} Tw^{-1} + (\alpha_3/C_{\text{max}})rw \leq 1 \\
& \quad (1/T_{\text{max}})T \leq 1, \quad T_{\text{min}} T^{-1} \leq 1 \\
& \quad (1/r_{\text{max}})r \leq 1, \quad \alpha_2 r \leq 1 \\
& \quad (1/w_{\text{max}})w \leq 1, \quad w_{\text{min}} w^{-1} \leq 1 \\
& \quad 10w^{-1} \leq 1.
\end{align*}$$

This is equivalent to the GP

$$\begin{align*}
\text{minimize} & \quad (1/\alpha_4)T^{-1}r^{-2} \\
\text{subject to} & \quad \alpha_1/C_{\text{max}} Tw^{-1} + (\alpha_3/C_{\text{max}})rw \leq 1 \\
& \quad (1/T_{\text{max}})T \leq 1, \quad T_{\text{min}} T^{-1} \leq 1 \\
& \quad (1/r_{\text{max}})r \leq 1, \quad \alpha_2 r \leq 1 \\
& \quad (1/w_{\text{max}})w \leq 1, \quad w_{\text{min}} w^{-1} \leq 1 \\
& \quad 10w^{-1} \leq 1.
\end{align*}$$
4.43 Eigenvalue optimization via SDP. Suppose $A : \mathbb{R}^n \to \mathbb{S}^n$ is affine, i.e.,

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$$

where $A_i \in \mathbb{S}^n$. Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x)$ denote the eigenvalues of $A(x)$. Show how to pose the following problems as SDPs.

(a) Minimize the maximum eigenvalue $\lambda_1(x)$.

(b) Minimize the spread of the eigenvalues, $\lambda_1(x) - \lambda_m(x)$.

(c) Minimize the condition number of $A(x)$, subject to $A(x) \succ 0$. The condition number is defined as $\kappa(A(x)) = \lambda_1(x)/\lambda_m(x)$, with domain $\{x \mid A(x) \succ 0\}$. You may assume that $A(x) \succ 0$ for at least one $x$.

Hint. You need to minimize $\lambda / \gamma$, subject to

$$0 < \gamma I \preceq A(x) \preceq \lambda I.$$ 

Change variables to $y = x / \gamma$, $t = \lambda / \gamma$, $s = 1 / \gamma$.

(d) Minimize the sum of the absolute values of the eigenvalues, $|\lambda_1(x)| + \cdots + |\lambda_m(x)|$.

Hint. Express $A(x)$ as $A(x) = A_+ - A_-$, where $A_+ \succeq 0$, $A_- \succeq 0$.

Solution.

(a) We use the property that $\lambda_1(x) \leq t$ if and only if $A(x) \preceq tI$. We minimize the maximum eigenvalue by solving the SDP

$$\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad A(x) \preceq tI.
\end{align*}$$

The variables are $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

(b) $\lambda_1(x) \leq t_1$ if and only if $A(x) \preceq t_1 I$ and $\lambda_m(A(x)) \geq t_2$ if and only if $A(x) \succeq t_2 I$,

so we can minimize $\lambda_1 - \lambda_m$ by solving

$$\begin{align*}
\text{minimize} & \quad t_1 - t_2 \\
\text{subject to} & \quad t_2 I \preceq A(x) \preceq t_1 I.
\end{align*}$$

This is an SDP with variables $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$, and $x \in \mathbb{R}^n$. 