Problem Set 8 – Due Thursday, November 29, 2012

This is a crucial problem set; with it, you should be coming to understand one of the most important concepts in this class: reductions.

With this problem set, you are back to doing problem sets on your own.

Problem 1. Classify each of the following languages as either recursive, or r.e. but not not co-r.e., or co-r.e. but not r.e., or neither r.e. nor co-r.e. (For ease of grading, please use these four labels.) You should be able to prove all of your claims, but, to keep things short, please provide a proof only for problems marked with a star. Proofs that a language is not r.e. or not co-r.e. must take the form of a reduction.

A \{\langle M \rangle : M \text{ is a TM that accepts some string of prime length}\}.

B\* \{\langle M, k \rangle : M \text{ is a TM that accepts at least one string of length } k\}.

C \{\langle M \rangle : M \text{ is a TM and } M \text{ has 100 states}\}.

D \{\langle M \rangle : M \text{ is a TM and } L(M) = L(M)^*\}.

E \{\langle M \rangle : M \text{ is a TM and } L(M) = \emptyset\}.

F \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is r.e.}\}.

G\* \{\langle M, k \rangle : M \text{ is a TM that runs forever (loops) on at least one string of length } k\}.

H \{\langle M \rangle : M \text{ is a C-program that halts on } \langle M \rangle\}.

I\* \{\langle M, k \rangle : M \text{ is a TM that accepts a string of length } k \text{ and diverges on a string of length } k\}. \text{ Assume that the underlying alphabet has at least two characters.}

J \{\langle M \rangle : M \text{ is a TM and } M \text{ will visit state } q_{20} \text{ when run on some input } x\}.

K\* \{\langle M, w \rangle : M \text{ is a TM and } M \text{ that uses at most 20 tape cells when run on } w\}.

L \{\langle G \rangle : G \text{ is a CFG and } G \text{ accepts an odd-length string}\}.

M \{\langle G_1, G_2 \rangle : G_1 \text{ and } G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}.

N\* \{\langle M \rangle : M \text{ is a TM that accepts some palindrome}\}.

Problem 2 Say that a language \(L = \{x_1, x_2, \ldots\}\) is enumerable if there exists a two-tape TM \(M\) that outputs \(x_1 x_2^2 x_3^2 \cdots\) on a designated output tape. The other tape is a designated work tape, and the output tape is write-only, with the head moving only from left-to-right. Say that \(L\) is enumerable in lexicographic order if \(L\) is enumerable, as above, and, additionally, \(x_1 < x_2 < x_3 < \cdots\), where “<” denotes the usual lexicographic ordering on strings.

\(^1\)Will count as more than one problem.
A. Prove that $L$ is r.e. iff $L$ is enumerable. (This explains the name “recursively enumerable.”)

B. Prove that $L$ is recursive iff it is enumerable in lexicographic order.

**Problem 3** Prove or disprove each of the following claims.

A. $A \leq_m A$.

B. If $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$.

C. If $A \leq_m B$ then $\overline{A} \leq_m \overline{B}$.

D. If $A$ is r.e. and $A \leq_m \overline{A}$ then $A$ is recursive.

E. If $A$ is recursive, then $A \leq_m a^*b^*$.

F. If $A \leq_m B$ then $B \leq_m A$.

G. If $A \leq_m B$ and $B \leq_m A$ then $A = B$. 