Practice Midterm Exam

Instructions: Relax. Smile. Be happy. Then think about each question for a few minutes before writing down a brief, correct answer!

Bon courage!
— Phil Rogaway

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E-mail:

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1 Short Answer [40 points]

(1) Draw a DFA that accepts $L = \{x \in \{a, b\}^* : x$ starts and ends with different characters$\}$. 

(2) Using the procedure we have seen in class, convert the following NFA into a DFA that accepts the same language.
(3) You are given a regular expression $\alpha$. Describe a decision procedure (algorithm) that determines if $L(\alpha)$ contains an odd-length string.

(4) Using the construction given in class and in your text, convert the regular expression $\alpha = (01 \cup 1)^*$ into an NFA for the same language.
(5) Complete the definition (as given in class or your book): A context free grammar is a 4-tuple $G = (V, \Sigma, R, S)$ where\footnote{Don’t just tell me what $V, \Sigma, R, S$ are called; tell me what they are, mathematically.}

(6) Use closure properties to show that $L = \{0^i1^j2^j : i, j \geq 0\}$ is not regular.
(7) Let $REG$ be the language of all (fully parenthesized) regular expressions over the alphabet \{0, 1\}.

Thus sample strings in $REG$ are:

$\varepsilon$

1

$((0 \circ 1) \cup 1)$

Prove that $REG$ is CF by giving a CFG for it.

(8) With $REG$ as defined above, prove that $REG$ is not regular by using the pumping lemma.
2 Justified True or False [35 points]

Put an X through the correct box. Where it says “Explain” provide a brief (but convincing) justification. No credit will be given to correct answers that lack a proper justification. Where appropriate, make your justification a counter-example. Throughout, we use $L$ and $R$ to denote languages.

1. If $\overline{L}$ is finite then $L$ is regular.
   Explain:
   True  False

2. Every regular language can be accepted by an NFA that has exactly 1,000,000 states.
   Explain:
   True  False

3. Let $R$ be regular and let $L$ is a subset of $R$. Then $L$ is regular.
   Explain:
   True  False
4. If $L$ is finite then $L^*$ is infinite.  
   Explain:

5. Suppose $L$ has the following property: for some number $N$, every string $s \in L$ having length at least $N$ can be partitioned into $s = xyz$ such that $|y| \geq 1$ and $xy^iz \in L$ for all $i \geq 0$. Then $L$ is regular.  
   Explain:

6. Every CFL $L$ can be generated by a CFG $G$ in which every rule $A \rightarrow \alpha$ satisfies $|\alpha| \leq 2$.  
   Explain:

7. For $L$ regular, let $d(L)$ be the number of states in a smallest DFA for $L$, and let $n(L)$ be the number of states in a smallest NFA for $L$. Then for any regular language $L$, $n(L) \leq (d(L))^2$.  
   Explain:
3 A Closure Property of Regular Languages [25 points]

If $L$ is a language over an alphabet $\Sigma$ let

$$\text{Two-Less}(L) = \{y \in \Sigma^* : \text{for some string } x \text{ having } |x| \leq 2, xy \in L\}.$$

**Part A.** (A warm-up, just to make sure you understand the definition.) Is one a subset of the other: $L$ and $\text{Two-Less}(L)$? If so, which is a subset of which?

**Part B.** (A warm-up, just to make sure you understand the definition.) Let $P = \{1^2, 1^3, 1^5, 1^7, 1^{11}, 1^{13}, \ldots\}$ be the set of prime numbers, encoded in unary. What’s the shortest string in $1^*$ which is not in $\text{Two-Less}(P)$?

**Part C.** (Now the main problem. This has nothing to do with Parts A or B.) Prove that if $L$ is regular, than so is $\text{Two-Less}(L)$. (Describe any construction you use both in clear English and by a formal definition.)