Practice Final Solutions (Fall 1995 Final)

1 Recall . . .

[30 points]

A. Complete the following definition:

Let $A, B \subseteq \{0,1\}^*$. We say that $A$ polynomial-time mapping reduces to $B$, written $A \leq_P B$, if . . .

there exists a polynomial-time computable function $f$ such that $x \in A$ iff $f(x) \in B$.

B. Complete the following definition:

A language $L$ is NP-Complete if . . .

(1) $L \in \text{NP}$, and
(2) for all $A \in \text{NP}$, $A \leq_P L$.

C. State the Cook/Levin Theorem:

Theorem [Cook/Levin]:

There is an NP-Complete language. In fact, SAT is NP-complete.

D. Complete the following statement of the pumping lemma for context free languages:

Theorem. If a language $L$ is context free then there exists a number $K$ such that . . .

for all $s \in L$ such that $|s| \geq K$ there exists $u, v, x, y, z$ such that $s = uvxyz$ and $|vxy| \leq K$ and $|vy| \geq 1$ and $uv^i xy^i z \in L$ for all $i \geq 0$.

E. In a sentence or two, state the “Church-Turing Thesis:"

Turing machines exactly capture our intuitive notion of what is effectively computable.

F. State Rice’s Theorem (which says that a certain class of languages is undecidable).

Theorem [Rice]:

If $P$ is a non-trivial property of the r.e. languages then $\{\langle M \rangle : L(M) \text{ has property } P \}$ is undecidable.

2 True/False [20 points]

Mark the correct box by putting an “X” through it. No justification required.

1. For every $i$, the language $L_i = \{a^i b^i c^i\}$ is context free. True

2. Assume $L$ is a regular language and let $L_{1101}$ be the subset of $L$ which contains the strings that end in a 1101. Then $L_{1101}$ is regular. True

3. If $L^*$ is decidable then $L$ is decidable. False

4. Every enumerable language can be accepted by a TM whose head only moves to the right. False
5. For any language $L$, the language $L^*$ is infinite. **False**

6. Let $\langle M \rangle$ be the encoding of a Turing machine $M$. Let $P(\langle M \rangle) = 0$ if $M$ on $\varepsilon$ halts in an even number of steps, 1 otherwise. Then $P$ satisfies the condition of Rice’s theorem: it is a nontrivial property of enumerable languages. **False**

7. The language $L = \{\langle M \rangle : L(M) \in \text{NP}\} \in \text{NP}$. **False**

8. Suppose $3\text{SAT} \leq_P L$ and $L \in \text{P}$. Then $P = \text{NP}$. **True**

9. $A_{\text{TM}}$ is NP-complete. **False**

10. If $L_1 \leq_P L_2$ and $L_2 \leq_P L_1$, then $L_1 = L_2$. **False**

### 3 Language Classification. [20 points]

Classify as:

- decidable
- r.e.
- co-r.e.
- neither
- enumerable but not decidable
- co-enumerable but not decidable
- neither enumerable nor co-enumerable

No explanation is required.

1. $\{\langle M \rangle : M$ is a TM which accepts some palindrome$\}$. r.e.
2. $\{\langle M \rangle : M$ is a TM which accepts some string of length $\geq |\langle M \rangle|\}$. r.e.
3. $\{d :$ the digit $d$ appears infinitely often in the decimal expansion of $\pi = 3.14159 \cdots\}$. decidable
4. $\{\langle G \rangle : G$ is a regular grammar and $L(G)$ contains an even-length string$\}$ decidable
5. $\{\langle G \rangle : G$ is a CFG and $L(G) = \Sigma^*\}$ co-r.e.
6. $\{\langle G, k \rangle : G$ is a graph containing no clique of size $k\}$. decidable
7. A language $L$ for which $L_{\Sigma^*} = \{\langle M \rangle : L(M) = \Sigma^*\} \leq_m L$. neither
8. $\{t : t$ is an instance of the tiling problem for which you can tile the plane using 10 or fewer tile types.$\}$. Omit this; we did not cover the tiling problem this term.

### 4 A Tight Bound on DFA Size [20 points]

Let $L_5 = \{111\}$ be the language over $\{0, 1\}$ which contains only the string 111.

(A) **Show that $L_5$ can be recognized by an 5-state DFA.**

Just draw the five-state DFA that accepts $L_5$.

(B) **Prove that $L_5$ can not be recognized by a 4-state DFA.**

Assume for contradiction that there exists a four state DFA $M = (Q, \Sigma, \delta, q_0, F)$ that accepts $L_5$. Consider the five states $q_i = \delta^*(q_0, 1^i)$ for $0 \leq i \leq 4$. We claim that all five of these states must be distinct, contradicting the pigeonhole principle. To see this, note that if $q_i = q_I$ for $0 \leq i < I \leq 4$ then $\delta^*(q_0, 1^i) = \delta^*(q_0, 1^I)$ so $\delta^*(q_0, 1^{i+1} - i)$ so $\delta^*(q_0, 1^I) = \delta^*(q_0, 1^{I+3-i})$ which is absurd, because the first state must be final and the second state must not be.
5 A Decision Procedure [20 points]

If \( \alpha \) is a regular expression, we write \( \alpha^2 \) for the regular expression \( \alpha \alpha \). Show that the following language is decidable (i.e., exhibit a decision procedure for this language):

\[
L = \{ \langle a, b, c \rangle : a, b \text{ and } c \text{ are regular expressions and } a^2 \cup b^2 = c^2. \}
\]

Using the procedure given in class, convert the regular expression \( aa \cup bb \) into an NFA, and then a DFA, \( M_1 \). Using the procedure given in class, convert the regular expression \( cc \) into an NFA, and then a DFA, \( M_2 \). Using the product construction, construct a DFA \( M \) the language of which is \( L(M_1) \oplus L(M_2) \). Using the procedure given in class, look to see if \( L(M) \) is empty. If it is, answer yes; otherwise, answer no.

6 Mapping Reductions [20 points]

Recall that, if \( w = a_1 \cdots a_n \in \Sigma^n \) is a string, \( w^R = a_n \cdots a_1 \) is the “reversal” of \( w \). If \( L \subseteq \Sigma^* \) is a language, \( L^R = \{ w^R : w \in L \} \). Let

\[
A_{TM} = \{ \langle M, w \rangle : M \text{ accepts } w \} \\
A_R = \{ \langle M \rangle : L(M) = (L(M))^R \}
\]

A. Show that \( A_{TM} \leq_m A_R \).

We must map \( \langle M, w \rangle \) into \( \langle M' \rangle \) such that \( M \) accepts \( w \) iff \( L(M') = (L(M'))^R \). The map must be Turing-computable. So let \( M' \) on input \( x \) behave as follows:

- If \( x = 01 \) then accept.
- Run \( M \) on \( w \)
- If \( M \) accepts \( w \), then accept
- If \( M \) rejects \( w \), then reject

Now if \( M \) accepts \( w \) then \( L(M') = \Sigma^* \) so \( L(M') = (L(M'))^R \); while if \( M \) does not accept \( w \) then \( L(M') = \{01\} \) so \( L(M') \neq (L(M'))^R \).

B. Show that \( \overline{A_{TM}} \leq_m A_R \).

We must map \( \langle M, w \rangle \) into \( \langle M' \rangle \) such that (a) if \( M \) does not accept \( w \) then \( L(M') = (L(M'))^R \), and (b) if \( M \) does accept \( w \) then \( L(M') \neq (L(M'))^R \). The map must be Turing-computable. So let \( M' \) on input \( x \) behave as follows:

- Run \( M \) on \( w \)
- If \( M \) accepts \( w \) and \( x = 01 \) then accept
- Reject

Now if \( M \) does not accept \( w \) then \( L(M') = \emptyset \) so \( L(M') = (L(M'))^R \); while if \( M \) does accept \( w \) then \( L(M') = \{01\} \) so \( L(M') \neq (L(M'))^R \).
7 NP-Completeness

Let $G = (V, E)$ be a graph. We say that $G' = (V', E')$ is a vertex-induced subgraph of $G$ if $V' \subseteq V$ and $E'$ is all the edges of $G$ both endpoints of which are in $V'$. Now suppose each edge $e \in E$ as an integer weight, $w(e)$. Then the weight of the subgraph $G'$ is just $\sum_{e' \in E'} w(e')$.

Show that the following language of graphs with heavy vertex-induced subgraphs is NP-Complete.

$$HVIS = \{ \langle G, w, B \rangle : G = (V, E) \text{ is a graph, } w : E \rightarrow \mathbb{Z} \text{ specifies an integer weight, } w(e), \text{ for each } e \in E, \text{ and } B \in \mathbb{Z} \text{ is an integer; and } G \text{ has some vertex-induced subgraph of weight at least } B. \}$$

First we show that HVIS $\in$ NP. Given $\langle G, w, B \rangle$, just “guess” a subset of vertices $V' \subseteq V$ and verify that the total weight of the subgraph induced by $V'$ is at least $B$. That is, the certificate identifies the induced subgraph.

Now we show that CLIQUE $\leq_P$ HVIS. Given a graph $G = (V, E)$ and a number $k$ we must map it into a graph $G' = (V', E')$ and a weight function $w$ and a bound $B$ such that $G$ has a clique of size $k$ if and only if $G'$ has an induced subgraph of weight at least $B$, relative to $w$. So given $G = (V, E)$ and $k$, where $n = |V|$, let $G' = (V, E')$ be the complete graph on $n$-node. Let $w(e)$ to be 1 if $e \in E$ and $w(e) = -n^2$ otherwise. Let $B = \binom{k}{2} = k(k - 1)/2$. Then if $G$ has a $k$-clique $W$ then $W$ names a vertex-induced subgraph of weight $\binom{k}{2}$ and so the instance of HVIS we have generated is a yes-instance. Conversely, if $G'$ has a vertex-induced subgraph of weight $B = \binom{k}{2}$ then the vertices of this subgraph must comprise a clique of size at least $k$ because it can contain no non-edges of $E$, as even one such edge would result in the weight of the induced subgraph being negative. The transformation is clearly polynomial time, so we are done.