Example 1: \(1 + 2 + 3 + \ldots + 100\)
\(1 + 2 + 3 + \ldots + n\)

\[S(n) = \sum_{i=1}^{n} i\]

\[S(n) = 1 + 2 + 3 + \ldots + (n-1) + n\]
\[\frac{n + (n-1) + (n-2) + \ldots + 2 + 1}{n+1} + (n+1) + (n+1) + \ldots + (n-1) + (n+1)\]

\[S(n) = \frac{n(n+1)}{2}\]
\[= \frac{100 \cdot 101}{2} = 50 \cdot 10 = 5050\]

Example 2: \(\sqrt{2}\) is an irrational number.

Definition: A real number \(x\) is rational if \(x = \frac{a}{b}\) for some \(a, b \in \mathbb{Z}\), \(b \neq 0\).

Definition: A real number \(x\) is irrational if it is not rational.

Assume for contradiction that \(x\) is rational:

\[\exists a, b \in \mathbb{Z}\] and \(a, b\) have no common division

\[\sqrt{2} = \frac{a}{b}\]
\[b \neq 0\]

If it is not true, \(2 = \frac{a^2}{b^2}\), \(2b^2 = a^2\)

\[a^2\] is even then, \(a\) is even and \(b\) is odd.
a even \( a = 2\alpha \) for \( \alpha \in \mathbb{Z} \)

b = \( 2\beta + 1 \) for some \( \beta \in \mathbb{Z} \)

\[
(2\beta + 1)^2 = 2\alpha^2
\]

\[
y\beta + y\beta + 1 = 2\alpha^2
\]

\[
y(\beta^2 + \beta) + 1 = 2\alpha^2
\]

\[\uparrow \quad \uparrow\]

odd \quad even

(uz it says t)

Example 3:

\[
\begin{array}{c}
\Box \\
\Box \\
\Box
\end{array}
\]

\[
A \quad B \quad C
\]

We need to move — from A to B or C on ring at a time
and big ring can’t go on small

Let \( T_n = \) moves \( \min \) \# of moves to move the \( n \) rings from one
key to another

Algorithm:

\[
T_n \leq T_{n-1} + 1 + T_{n-1}
\]

\[
\leq 2T_{n-1} + 1 \quad \text{if} \quad n \geq 2
\]

\[
T_1 = 1 \quad \text{if} \quad n = 1
\]

\[
n = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \text{formula}
\]

\[
= 1 \quad 3 \quad 7 \quad 15 \quad 31 \quad \text{guess}
\]
Fix any scheme that moves the \( n \) rings from one peg to another.

Let's name Scheme "M".

After a lot of shuffle we got the bottom ring on \( C \) on top.

M moved the biggest ring from a peg to \( C \) for the last time.

- M spent \( \geq T_{n-1} \) moves to clear the \( n-1 \) rings off of the biggest ring.
- M spent \( \equiv T_{n-1} \) moves to transport the final \( n-1 \) ring to \( C \).

\[
T_{n-1} \equiv 2T_{n-1} + 1
\]

\[
T_n = 2T_{n-1} + 1
= 2(2T_{n-2} + 1) + 1
= 2^2 T_{n-2} + 1 + 2
= 2^2 (2T_{n-3} + 1) + (1+2)
= 2^3 T_{n-3} + 1 + 2 + 2^2
= 2^4 T_{n-4} + (1 + 2 + 2^2 + 2^3)
= 1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1}
\]
Example Card Shuffle:

1 2 3 4 5 6 7 8

We need ideas to solve big puzzles and that idea called Rising Sequence

So we have:

4 1 5 6 2 7 8 3

Sequence in main number order:

2 4 1 7 8 5 3 6

Rising Sequence: A maximal consecutive subsequence of numbers

Sequence “2, 4, 6, 8, 10”

1 2 3 ..... 52 1 rising sequence

52 53 54 ..... 2 1 62 rising sequence

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