Today:

- Probability

Announcements:
- Please work out the old final for Thursday
- Final’s week OH are online. Unfortunately, I am out of town. I will check the chat room as often as I can.
- Final: you may not leave during the last 30 mins.

**Poker examples and first use of probability**

Let’s introduce probability informally

- **straight flush** = five consecutive cards: A2345, 23456, 34567, ..., 89AJQ, 9AJQK, AJQKA in any suit. So 40 possible.
- **royal flush** = AJQKA of one suit. So 4 possible
- **four of a kind** = four cards of one value, eg., four 9’s
- **full house** = 3 cards of one value, 2 cards of another value. Eg, three 10’s and two 4’s.
- **flush** = five cards of a single suit
- **three of a kind** = three cards of one value, a fourth card of a different value, and a fifth card of a third value
- **two pairs** = two cards of one value, two more cards of a second value, and the remaining card of a third value
- **one pair** = two cards of one value, but not classified above

1. **How many poker hands** are there?

   Answer: \( C(52,5) = 2,598,960 \)

2. **How many poker hands are full houses?**

   Answer: A full house can be partially identified by a pair, like (J,8), where the first component of the pair is what you have **three** of, the second component is what you have **two** of. So there are \( P(13,2) = 13 \times 12 \) such pairs. For each there are \( C(4,3) = 4 \) ways to choose the first component, and \( C(4,2) = 6 \) ways to choose the second component. So all together there are

   \[ 13 \times 12 \times 4 \times 6 = 3,744 \text{ possible full houses.} \]

   The probability of being dealt a full house is therefore

   \[ 3,744 / 2,598,960 \approx 0.001441 \approx 0.14 \% \]

   \[ P[\text{FullHouse}] \approx 0.001441 \]
The probability of an event is a real number between 0 and 1 (inclusive). If asked what’s the probability of something, don’t answer with a “percent”, and don’t answer with something outside of [0,1]. When we give something in “percent’s”, we are giving a probability multiplied by 100.

3. How many poker hands are two pairs?

Answer: We can partially identify two pairs as in {J, 8}. Note that now the pair is now unordered. There are $C(13,2)$ such sets. For each there are $C(4,2)$ ways to choose the larger card and $C(4,2)$ ways to choose the smaller card. There are now 52 – 8 remaining cards one can choose as the fifth card (to avoid a full house, there are 8 “forbidden” cards). So the total is

$$C(13,2)*C(4,2)*C(4,2)*44 = 123,552.$$ 

The chance of being dealt two pairs is therefore

$$C(13,2)*C(4,2)*C(4,2)*44 / C(52,5) = 123,552/2,598,960 \approx 0.047539 \approx 4.75\%$$

$P[TwoPairs] \approx 0.047539$

**Basic definitions / theory**

Schaum’s, chapter 7.
Probability does not appear at all in Biggs.

**Def:** A (finite) **probability space** $(S,P)$ is
- a finite set $S$ (the sample space) and
- a function $P: S \rightarrow [0,1]$ (the probability measure) such that that

$$\sum_{x \in S} P(x) = 1$$

(alternative notation: $\omega, \mu, \Omega$ for $x, P, S$)

In general, whenever you hear probability make sure that you are clear what is the probability space is: what is the sample space and what is the probability measure on it.

An **outcome** is a point in $S$.

**Def:** Let $(S, P)$ be a probability space.

An **event** is a subset of $S$.

**Def:** Let $A$ be an event of probability space $(S, P)$.

$$P(A) = \sum_{a \in A} P(a)$$  (I’m used to using Pr, will probability slip)

*The probability of event A. By convention, $P(\emptyset)=0$*

**Def:** The uniform distribution is the one where $P(a) = 1/|S|$ —ie, all points are equiprobable.

**Def:** Events $A$ and $B$ are independent if $P(A \cap B) = P(A) P(B)$. 
**Def:** A **random variable** is a function $X : S \rightarrow \mathbb{R}$ from the sample space to the reals.

**Def:** $E[X] = \sum_{s \in S} P(s) X(s)$ // expected value of $X$ ("average value")

**Def:** if $B \neq \emptyset$ then $P(A | B) = \frac{P(A \cap B)}{P(B)}$

**Propositions:**
- $P(\emptyset) = 0$ // by definition
- $P(S) = 1$
- $P(A) + P(A^c) = 1$, or $P(A) = 1 - P(A^c)$

- If $A$ and $B$ are disjoint events (that is, disjoint sets) then
  $P(A \cup B) = P(A) + P(B)$

- ("sum bound")
  $P(A \cup B) \leq P(A) + P(B)$

- In general,
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ // inclusion-exclusion principle

- If $B_1, B_2$ disjoint, nonempty events whose union is $S$ then
  $P(A) = P(A | B_1) P(B_1) + P(A | B_2) P(B_2)$

- $E(X+Y) = E(X) + E(Y)$ // expectation is linear.

**Eg 1: Dice**
The singular, the students assure me, is **die**. Like **mice** and **mie**. I guess.

- You roll a fair die six times:
  $S = \{1,2,3,4,5,6\}$
  $P(1) = P(2) = \ldots = P(6) = 1/6$

  "you roll an even number" is an event.
  Event is $A = \{2,4,6\}$. $P(A) = 3 \times (1/6) = 1/2$.

- Pair of dice.
  $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$
  $P((a,b)) = 1/36$ for all $(a,b)$ in $S$

Illustrate independence.

$P(\text{left die even and right die even}) = P(\text{left die even}) P(\text{right die even})$
$= (1/2) (1/2) = 1/4$

- Pair of dice, what's the chance of rolling an "8"?

Event $E = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$
$P(E) = 5/36$
Be careful: $P(E) = |E|/|S|$ only if we are assuming the uniform distribution.

- Pair of dice, what’s the chance of rolling an "8" if I tell you that both numbers came out even?
  
  **Method 1:** Imagine the new probability space:
  
  $$(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)$$

  So probability is $3/9 = 1/3$

  **Method 2:** A little more mechanically
  
  $A = "rolled an 8"
  
  $B = "both die are even"

  $$P(A | B) = P(A \cap B)/P(B)$$

  $$= (3/36) / (9/36) = 1/3$$

**Eg 2. Back to the Poker examples**

What’s the probability space?
Sample space has $|S| = C(52,5)$
We can regard the points of $S$ as 5-elements subsets
of $\{2C,2D,2H,2S, 3C,3D,3H,3S,\ldots, KC,KD,KH,KS, AC,AD,AH,AS\}$ or $\{1, \ldots, 52\}$
Probability measure is uniform: $P(a) = 1/|S|$.

**Eg 3: Fair coin**

Flip a fair coin 100 times.
What is the probability space?
$S = \{0,1\}^{100}$
$P(s) = 2^{-100}$ for all $s$ in $S$.

What is the chance of getting exactly 50 out of the 100 coin flips land heads?

$$P(50\text{Heads}) = C(100,50) / 2^{100} \approx 0.07959$$  // note “100 choose 50” to Google
$$P(51\text{Heads}) = C(100,51) / 2^{100} \approx 0.07803$$

**Eg 4: Biased coin**

Now, what if the coin is biased?
Say that the coin lands heads with probability $p= .51$ and tails with probability $1-p= .49$.
each flip independent of the rest.

You flip the unfair coin 100 times. The coin lands heads a fraction $p=0.51$ of the time:
\[ S = \{0,1\}^{100} \quad \text{(same as before, but now)} \]
\[ P(x) = p^{\#1(x)} (1-p)^{\#0(x)} \quad \text{where } \#1(x) = \text{the number of 1-bits in the string } x \text{ and} \]
\[ \#0(x) = \text{the number of 0-bits in the string } x. \]

What's the Probability of 50 and 51 heads now?

\[ P(50\text{Heads}) = \binom{100}{50}(.51^{50})(.49^{50}) \approx 0.07801 \]
\[ P(51\text{Heads}) = \binom{100}{51}(.51^{51})(.49^{49}) \approx 0.07906 \]

Makes sense -- 51 heads should now be the most likely number, and things should fall off from there. Before, 50 heads was the most likely outcome.

**Eg 5: Birthday phenomenon**

\( n = 23 \) people gather in a room.
What' the chance that some two have the same birthday?
Assume nobody born 2/29, all other birthdays equiprobable.
\[ S = [1..365]^{23} \]

\[ P(\text{SameBirthday}) = 1 - \Pr(\text{AllBirthdaysDifferent}) \]
\[ = 1 - (1-1/365)(1-2/365) ... (1-22/365) \]
\[ = 1 - \prod_{i=1}^{22} (1-1/i) \approx 0.507 \]

That’s as far as we got in lecture. See
http://en.wikipedia.org/wiki/Birthday_problem#Calculating_the_probability
if you didn't follow.