Today:

- A last bit of probability
- “Review session”
- Closing comments
- You grade me

Announcements:

- eval.ucdavis.edu ← for course evaluations
- special office hours online
- Final: you may not leave during the last 30 mins.

**Basic definitions / theory**

**Review:**

**Def:** A (finite) **probability space** \((S, P)\) is

- a finite set \(S\) (**the sample space**)
- a function \(P : S \rightarrow [0,1]\) (**the probability measure**) such that that

\[
\sum_{x \in S} P(x) = 1
\]

(alternative notation: \(\omega, \mu, \Omega\) for \(x, P, S\))

In general, whenever you hear **probability** make sure that you are clear **what** is the probability space is: what is the sample space and what is the probability measure on it.

An **outcome** is a point in \(S\).

**Def:** Let \((S, P)\) be a probability space.

An **event** is a subset of \(S\).

**Def:** Let \(A\) be an event of probability space \((S, P)\).

\[
P(A) = \sum_{a \in A} P(a)
\]

(I’m used to using Pr, will probability slip)

*The probability of event \(A\). By convention, \(P(\emptyset) = 0\)*

**Def:** The **uniform** distribution is the one where \(P(a) = 1/|S|\) —ie, all points are equiprobable.

**Def:** Events \(A\) and \(B\) are **independent** if \(P(A \cap B) = P(A) P(B)\).

**Def:** if \(B \neq \emptyset\) then \(P(A|B) = P(A \cap B) / P(B)\)

**Prop:** \(P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)\)
Def: A **random variable** is a function \( X: S \to \mathbb{R} \) from the sample space to the reals.

Def: \( \mathbb{E}[X] = \sum_{s \in S} P(s) X(s) \)  // expected value of \( X \) (“average value”)

- Prop: \( \mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) \)  // expectation is linear.

**Eg 1: Alice rolls a die.**

What’s do you expect the **square** of her roll to be?

could be 1 .... could be a 36!

Definition: a RV is a function from X: S \( \to \) \( \mathbb{R} \)

Definition: \( \mathbb{E}[X] = \sum_{s} X(s) P(s) \)

So, in this problem,

\[
\mathbb{E}[X] = 1(1/6) + 2^2(1/6) + 3^2(1/6) + \ldots + 6^2(1/6)
\]
\[
= 1/6(1+4+9+15+25+36)
\]
\[
= 91/6
\]
\[
\approx 15.2
\]

Exercise: Repeat, supposing she rolls a *pair* of dice:

**Eg 2: Subway**

When Pablo, a CS grad student at MIT, leaves his office late at night, he wanders to the subway and takes the first train North or South:

There are trains every 10 mins, both N and S. Yet during the last 31 days, Pablo only has gone home only 3 times, and this seems to be about typical, in his experience. Explain what is going on and compute Pablo’s average wait time for a train? Assume trains arrive at the station on 1-minute intervals.
Example:

<table>
<thead>
<tr>
<th>Northbound</th>
<th>Southbound</th>
</tr>
</thead>
<tbody>
<tr>
<td>11:00</td>
<td>11:01</td>
</tr>
<tr>
<td>1 min</td>
<td>9 min</td>
</tr>
<tr>
<td>11:10</td>
<td>11:11</td>
</tr>
<tr>
<td>1 min</td>
<td>9 min</td>
</tr>
<tr>
<td>11:20</td>
<td>11:21</td>
</tr>
</tbody>
</table>

Let $X = \text{Wait time}$

$\frac{1}{10}(0.5 \text{ min}) + \frac{9}{10}(4.5 \text{ mins})$

$= 0.05 + 4.05 \text{ mins}$

$= 4.1 \text{ mins}$

How should the trains be staggered to minimize Pablo’s wait time?

| 11:00      | 11:05      |
| 11:10      | 11:05      |
| 11:20      | 11:20      |

Average wait time will be 2.5 mins
Ex 3: Monty Hall Problem
*Let's make a Deal* (1963-1968)

A good prize is hidden behind a random curtain/door (junk, “zonk”, behind the other two). You choose an arbitrary door. The host opens one of the unselected doors that does NOT contain the good prize. Should you switch to the other door?

```
<table>
<thead>
<tr>
<th>loc of good prize</th>
<th>which door to open if host must choose</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2, 3} \times {0, 1}</td>
<td>not relevant</td>
</tr>
</tbody>
</table>
```

WIN = get good prize

Strategy STICK: choose door 1 and stick with it:  $$P(WIN)=1/3$$

Strategy SWITCH: choose door 1 and then switch (always) when offered a chance.

- (1,0) Lose    Win
- (2,0) Win     Win
- (3,0) Win     Win

$$P(WIN) = \frac{4}{6} = \frac{2}{3}$$

Or: $$P(WIN) = P(WIN | initialCorrect) Pr(initialCorrect) + P(WIN | initialIncorrect) Pr(initialIncorrect)$$

$$= 0 + 1 \cdot \frac{2}{3} = \frac{2}{3}$$

Or just choose door 1
lose  win  win

Eg 4: Smaller / Bigger game
Alice uniformly chooses two distinct numbers between 1 and 10 and announces the first. Bob guesses if the second is smaller or bigger than the first. How should Bob play optimally—and, if he does so, what is his chance to win?

As usual, start by figuring out the sample space

\[ S = \{(i,j) \in \{1..10\}^2 : i \neq j\} \]

Alice announces any of

\[
1 2 3 4 5 6 7 8 9 10
\]

Bob's strategy:

- If Alice announces 1, 2, 3, 4, 5 guess **SMALLER**
- If Alice announces 6, 7, 8, 9, 10 guess **BIGGER**

\[
P(\text{Win}) = P(\text{Win} | \text{AliceAnswers1}) P(\text{AliceAnswers1}) + P(\text{Win} | \text{AliceAnswers2}) P(\text{AliceAnswers2}) + \ldots + P(\text{Win} | \text{AliceAnswers10}) P(\text{AliceAnswers10})
\]

\[
= (1/10) (P(\text{Win} | \text{AliceAnswers1}) + \ldots + P(\text{Win} | \text{AliceAnswers10})
\]

\[
\]

\[
= (1/10) (7*10/9) \quad \text{numbers clearly average 7}
\]

\[
= 70/90
\]

\[
= 7/9 \approx 78\
\]