Problem 1.

[Linz, Section 4.2, Exercise 6].

Before illustrating the algorithm, we need to prove that the family of regular languages are closed under reversal.

Let $L^R = \{w : w^R \in L\}$ and prove that $L^R$ is also regular:

Since $L$ is regular, it is accepted by a DFA $M = (Q, \Sigma, \delta, q_0, F)$.

Let $M_R = (Q \cup \{q_R\}, \Sigma, \delta_R, q_R, \{q_0\})$ be an NFA that accepts $L^R$, where:

- $q_R$ is the new start state. Let $\delta_R(q_R, \lambda) = F$.
- For each transition in $\delta$: $\delta(q_i, a) = q_j \implies \delta_R(q_j, a) = q_i$.
- $\{q_0\}$ is the set of final states for $M_R$.

You can prove by induction that $L(M_R) = L^R$ – i.e., $x \in L(M_R) \iff x \in L^R$.

The following describes an algorithm that determines whether a regular language $L$ contains any string $w$ such that $w^R \in L$ in finite steps:

1. Construct a DFA $M = (Q, \Sigma, \delta, q_0, F)$, where $L = L(M)$.
2. Construct a DFA $M_R$, where $L(M_R) = \{w : w^R \in L\}$.
3. Construct a DFA $M'$, such that $L(M') = L(M) \cap L(M_R)$ (based on Theorem 4.1).
4. If $L(M') \neq \emptyset$ (using the algorithm from Theorem 4.6 to determine this property), then there exists some $w^R \in L$. 
The following describes an algorithm that determines whether a regular language $L$ contains infinite number of even-length strings in finite steps:

1. Construct a DFA $M = (Q, \Sigma, \delta, q_0, F)$, where $L = L(M)$.
2. Construct a DFA $M_E$, where $L(M_E) = \{w : w \in \Sigma^* \text{ and } |w| \mod 2 = 0\}$.
3. Construct a DFA $M'$, such that $L(M') = L(M) \cap L(M_E)$ (based on Theorem 4.1).
4. If $L(M')$ is infinite (using the algorithm from Theorem 4.6 to determine this property), then $L$ contains infinite even-length strings.

No. Prove by contradiction using the pumping lemma:
Given $m$, let $w = a^{m!}2^m$ be decomposed into $xyz$, where $|xy| \leq m$ and $y \neq \lambda$. Suppose $y = a^k$, where $1 \leq k \leq m$, then we pump $i$ times to generate a string that contains $2^m + k \cdot (i - 1)$ $a$'s. Let $i = 2$, then $xy^2z = a^{2^m+k}$. Since $2^m + k < 2^{m+1}$, $a^{2^m+k} \notin L$. Thus, $L$ is not regular.

No. Prove by contradiction using the pumping lemma:
Given $m$, let $w = a^{p \cdot q}$, where $p$ and $q$ are prime numbers and $p \cdot q \geq m$, $w$ can be decomposed into $xyz$, where $|xy| \leq m$ and $y \neq \lambda$. Suppose $y = a^k$, where $1 \leq k \leq m$, then we pump $i$ times to generate a string that contains $p \cdot q + k \cdot (i - 1)$ $a$'s. Let $i = 1 + p \cdot q$, then $p \cdot q + k \cdot [(1 + p \cdot q) - 1] = p \cdot q + k \cdot p \cdot q = p \cdot q \cdot (k + 1)$. Since $p \cdot q \cdot (k + 1)$ cannot be a product of two primes, $a^{p \cdot q \cdot (k+1)} \notin L$. Thus, $L$ is not regular.

$L^* = \{a^n : n \geq 2, \text{ is the sum of primes}\} = \{a^n : n = 0 \text{ and } n \geq 2\}$. Since a simple DFA can be constructed for $L^*$, $L^*$ is regular.

Given $m$, let $w = a^{(m!)^2+1} \in L$. $w$ can be decomposed into $xyz$, where $|xy| \leq m$ and $y \neq \lambda$. Suppose $y = a^k$, where $1 \leq k \leq m$, then we pump $i$ times to generate a string with $(m!)^2 + 1 + k \cdot (i - 1)$ $a$'s. Let $i = 1 + \frac{2\cdot m!}{k}$. 

[The rest of the text contains exercises and proofs related to regular languages and DFAs, which are not repeated here for brevity.]
Then, \((m!)^2 + 1 + k \cdot [(1 + \frac{2m!}{k}) - 1] = (m!)^2 + 2(m!) + 1 = (m! + 1)^2\).

Thus, \(L\) is not regular.

[Linz, Section 4.3, Exercise 10(b).]

Example 4.11 (on page 119) shows that \(\mathcal{L}\) is not regular. Thus, by closure properties \(L\) is not regular.

[Linz, Section 4.3, Exercise 15(e).]

No. Prove by contradiction using the pumping lemma.

Given \(m\), let \(w = a^mb^m \in L\). \(w\) can be decomposed into \(xyz\), where \(|xy| \leq m\) and \(y \neq \lambda\). Since \(|xy| \leq m\), \(y\) contains only \(a\)'s. Suppose \(y = a^k\), where \(1 \leq k \leq m\), then we pump \(i\) times to generate a string with \(m + k \cdot (i - 1)\) \(a\)'s and \(m\) \(b\)'s. Let \(i = 2\), then \(xy^2z = a^{m+k}b^m\). Since \(m < m + k\), \(a^{m+k}b^m \notin L\). Thus, \(L\) is not regular.

[Linz, Section 4.3, Exercise 15(f).]

Yes. We can construct a DFA \(M = (Q, \Sigma, \delta, q_0, F)\) that accepts \(L\), where \(Q = \{q_i : 0 \leq i \leq 201\}\), \(\Sigma = \{a, b\}\), \(F = \{q_i : 100 \leq i \leq 200\}\), and \(\delta\) is defined as follows:

- \(\delta(q_{201}, a) = q_{201}\) and \(\delta(q_{201}, b) = q_{201}\)
- \(\delta(q_j, a) = q_{j+1}\) and \(\delta(q_j, b) = q_{201}\), for \(0 \leq j < 100\)
- \(\delta(q_{100}, a) = q_{100}\)
- \(\delta(q_j, b) = q_{j+1}\) and \(\delta(q_j, a) = q_{201}\), for \(100 \leq j < 200\)
- \(\delta(q_{200}, a) = q_{201}\) and \(\delta(q_{200}, b) = q_{201}\)

[Linz, Section 4.3, Exercise 24.]

No. For example, suppose \(L_1 = L(a^*b^*)\) and \(L_2 = \{a^n b^n : n \geq 0\}\). Clearly, \(L_1\) is regular, but \(L_2\) is not (shown in Example 4.7). However, \(L_1 \cup L_2 = L(a^*b^*)\) is regular.

[Linz, Section 4.3, Exercise 26.]

\(L\) is regular and can be accepted by a DFA similar to Section 4.3, Exercise 15(f).
[a.]
No. The pumping lemma is used for proof by contradiction. Although we could show that any pumped string is still in $L$, there is nothing in the pumping lemma that allows us to conclude that $L$ is regular.

[b.]
No. For any given value of $m$, there is always a $w$ such that $w_i \in L$ where $i \geq 0$. 