VII. Graph Algorithms
Notion of graphs

Basic terminology

- **Graph** $G = (V, E)$:
  - $V = \{v_i\}$ = set of **vertices**
  - $E = \text{set of edges} = \text{a subset of } V \times V = \{(v_i, v_j)\}$
Notion of graphs

Basic terminology

- Graph $G = (V, E)$:
  - $V = \{v_i\}$ = set of vertices
  - $E =$ set of edges = a subset of $\bigcup_{i \neq j} \{(v_i, v_j)\}$
  - $|E| = O(|V|^2)$
    - dense graph: $|E| \approx |V|^2$
    - sparse graph: $|E| \approx |V|$
    - If $G$ is connected, then $|E| \geq |V| - 1$. 

Reading: Appendix B.4, pp.1168-1172 of [CLRS,3rd ed.]
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Notion of graphs

Representing a graph by an Adjacency Matrix

- $A = (a_{ij})$ is a $|V| \times |V|$ matrix, where

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a_{ij} = \begin{cases} 
1, & \text{if } (v_i, v_j) \in E \\
0, & \text{otherwise}
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- If $G$ is undirected, $A$ is symmetric, i.e., $A^T = A$.
- $A$ is typically very sparse

  *use a sparse storage scheme in practice*
Notion of graphs

Representing a graph by an **Incidence Matrix**

$B = (b_{ij})$ is a $|V| \times |E|$ matrix, where

$$
b_{ij} = \begin{cases} 
1, & \text{if edge } e_j \text{ enters vertex } v_i \\
-1, & \text{if edge } e_j \text{ leaves vertex } v_i \\
0, & \text{otherwise}
\end{cases}$$
Notion of graphs

Representing a graph by an Adjacency List

- For each vertex $v$,

$$\text{Adj}[v] = \{ \text{vertices adjacent to } v \}$$
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- Variation: could also keep second list of edges coming into vertex.

Answer: $\Theta(|V| + |E|)$ ("sparse representation")
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Degree of a vertex

▶ undirected graph:

The degree of a vertex = the number of incident edges

\[ \sum_{v \in V} \text{degree}(V) = 2 |E| \]

The handshaking theorem.

▶ directed graph (digraph):

out-degree and in-degree

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  - out-degree and in-degree
  - total # of items in the adj. list = \( \sum_{v \in V} \text{out-degree}(V) = |E| \)