VII. Graph Algorithms
Notion of graphs

Basic terminology

- Graph \( G = (V, E) \):
  - \( V = \{v_i\} = \) set of vertices
  - \( E = \) set of edges = a subset of \( V \times V = \{(v_i, v_j)\} \)

- Dense graph: \( |E| \approx |V|^2 \)
- Sparse graph: \( |E| \approx |V| \)

- If \( G \) is connected, then \( |E| \geq |V| - 1 \).

- Some variants:
  - Undirected: edge \((u, v) = (v, u)\)
  - Directed: \((u, v)\) is edge from \( u \) to \( v \).
  - Weighted: weight on either edge or vertex
  - Multigraph: multiple edges between vertices

Reading: Appendix B.4, pp.1168-1172 of [CLRS, 3rd ed.]
Notion of graphs

Basic terminology

- **Graph** $G = (V, E)$:
  - $V = \{v_i\}$ = set of **vertices**
  - $E$ = set of **edges** = a subset of $V \times V = \{(v_i, v_j)\}$

- $|E| = O(|V|^2)$
  - **dense** graph: $|E| \approx |V|^2$
  - **sparse** graph: $|E| \approx |V|$
  - If $G$ is connected, then $|E| \geq |V| - 1$. 

阅读：Appendix B.4, pp.1168-1172 of [CLRS, 3rd ed.]
Notion of graphs

Basic terminology

- **Graph** $G = (V, E)$:
  - $V = \{v_i\} =$ set of vertices
  - $E =$ set of edges = a subset of $V \times V = \{(v_i, v_j)\}$

- $|E| = O(|V|^2)$
  - **dense graph**: $|E| \approx |V|^2$
  - **sparse graph**: $|E| \approx |V|$
  - If $G$ is connected, then $|E| \geq |V| - 1$.

- **Some variants**
  - **undirected**: edge $(u, v) = (v, u)$
Notion of graphs

Basic terminology

▶ Graph \( G = (V, E) \):
  ▶ \( V = \{v_i\} \) = set of vertices
  ▶ \( E \) = set of edges = a subset of \( V \times V = \{(v_i, v_j)\} \)

▶ \( |E| = O(|V|^2) \)
  ▶ dense graph: \( |E| \approx |V|^2 \)
  ▶ sparse graph: \( |E| \approx |V| \)
  ▶ If \( G \) is connected, then \( |E| \geq |V| - 1 \).

▶ Some variants
  ▶ undirected: edge \((u, v) = (v, u)\)
  ▶ directed: \((u, v)\) is edge from \(u\) to \(v\).
Notion of graphs

Basic terminology

- **Graph** $G = (V, E)$:
  - $V = \{v_i\} =$ set of vertices
  - $E =$ set of edges = a subset of $V \times V = \{(v_i, v_j)\}$

- $|E| = O(|V|^2)$
  - dense graph: $|E| \approx |V|^2$
  - sparse graph: $|E| \approx |V|$
  - If $G$ is connected, then $|E| \geq |V| - 1$

Some variants

- **undirected**: edge $(u, v) = (v, u)$
- **directed**: $(u, v)$ is edge from $u$ to $v$
- **weighted**: weight on either edge or vertex
Notion of graphs

Basic terminology

- **Graph** $G = (V, E)$:
  - $V = \{v_i\}$ = set of vertices
  - $E$ = set of edges = a subset of $V \times V = \{(v_i, v_j)\}$

- $|E| = O(|V|^2)$
  - **dense graph**: $|E| \approx |V|^2$
  - **sparse graph**: $|E| \approx |V|$
  - If $G$ is connected, then $|E| \geq |V| - 1$.

- Some variants
  - **undirected**: edge $(u, v) = (v, u)$
  - **directed**: $(u, v)$ is edge from $u$ to $v$.
  - **weighted**: weight on either edge or vertex
  - **multigraph**: multiple edges between vertices
Notion of graphs

Basic terminology

- **Graph** $G = (V, E)$:
  - $V = \{v_i\} =$ set of **vertices**
  - $E = $ set of **edges** = a subset of $V \times V = \{(v_i, v_j)\}$

- $|E| = O(|V|^2)$
  - **dense** graph: $|E| \approx |V|^2$
  - **sparse** graph: $|E| \approx |V|$
  - If $G$ is connected, then $|E| \geq |V| - 1$.

Some variants

- **undirected**: edge $(u, v) = (v, u)$
- **directed**: $(u, v)$ is edge from $u$ to $v$.
- **weighted**: weight on either edge or vertex
- **multigraph**: multiple edges between vertices

- Reading: Appendix B.4, pp.1168-1172 of [CLRS,3rd ed.]
Notion of graphs

Representing a graph by an **Adjacency Matrix** $A$

- $A = (a_{ij})$ is a $|V| \times |V|$ matrix, where

$$a_{ij} = \begin{cases} 
1, & \text{if } (v_i, v_j) \in E \\
0, & \text{otherwise} 
\end{cases}$$
Notion of graphs

Representing a graph by an Adjacency Matrix $A$

$A = (a_{ij})$ is a $|V| \times |V|$ matrix, where

$$a_{ij} = \begin{cases} 
1, & \text{if } (v_i, v_j) \in E \\
0, & \text{otherwise}
\end{cases}$$

If $G$ is undirected, $A$ is symmetric, i.e., $A^T = A$. 
Notion of graphs

Representing a graph by an Adjacency Matrix $A$

- $A = (a_{ij})$ is a $|V| \times |V|$ matrix, where

$$a_{ij} = \begin{cases} 
1, & \text{if } (v_i, v_j) \in E \\
0, & \text{otherwise}
\end{cases}$$

- If $G$ is undirected, $A$ is symmetric, i.e., $A^T = A$.

- $A$ is typically very sparse

  *use a sparse storage scheme in practice*
Notion of graphs

Representing a graph by an **Incidence Matrix** $B$

- $B = (b_{ij})$ is a $|V| \times |E|$ matrix, where

$$b_{ij} = \begin{cases} 
1, & \text{if edge } e_j \text{ enters vertex } v_i \\
-1, & \text{if edge } e_j \text{ leaves vertex } v_i \\
0, & \text{otherwise}
\end{cases}$$
Notion of graphs

Representing a graph by an **Adjacency List**

- For each vertex \( v \),

\[
\text{Adj}[v] = \{ \text{vertices adjacent to } v \}
\]
Notion of graphs

Representing a graph by an Adjacency List

- For each vertex $v$,
  \[ \text{Adj}[v] = \{ \text{vertices adjacent to } v \} \]
- Variation: could also keep second list of edges coming into vertex.

Answer: $\Theta(|V| + |E|)$ ("sparse representation")
Notion of graphs

Representing a graph by an Adjacency List

- For each vertex $v$,
  \[ \text{Adj}[v] = \{ \text{vertices adjacent to } v \} \]
- Variation: could also keep second list of edges coming into vertex.
- How much storage is needed?

Answer: $\Theta(|V| + |E|)$ ("sparse representation")
Notion of graphs

Representing a graph by an Adjacency List

- For each vertex \( v \),
  \[
  \text{Adj}[v] = \{ \text{vertices adjacent to } v \} 
  \]
- Variation: could also keep second list of edges coming into vertex.
- How much storage is needed?
  Answer: \( \Theta(|V| + |E|) \) ("sparse representation")
Notion of graphs

Degree of a vertex
Notion of graphs

Degree of a vertex

▶ undirected graph:
  ▶ The degree of a vertex = the number of incident edges

▶ directed graph (digraph):
  ▶ out-degree and in-degree
  ▶ \( \sum_{v \in V} \text{out-degree}(v) = \sum_{v \in V} \text{in-degree}(v) = |E| = \) total number of items in the adjacency list
Notion of graphs

Degree of a vertex

▶ undirected graph:
  ▶ The degree of a vertex = the number of incident edges
  ▶ The handshaking theorem:

\[ \sum_{v \in V} \text{degree}(V) = 2|E| \]

= total number of items in the adjacency list
Notion of graphs

Degree of a vertex

▶ undirected graph:
  ▶ The **degree** of a vertex = the number of incident edges
  ▶ The handshaking theorem:
    \[
    \sum_{v \in V} \text{degree}(V) = 2|E|
    \]
    = total number of items in the adjacency list

▶ directed graph (digraph):
  ▶ out-degree and in-degree
Notion of graphs

Degree of a vertex

▶ undirected graph:
  ▶ The degree of a vertex = the number of incident edges
  ▶ The handshaking theorem:

\[
\sum_{v \in V} \text{degree}(V) = 2|E|
\]

= total number of items in the adjacency list

▶ directed graph (digraph):
  ▶ out-degree and in-degree

\[
\sum_{v \in V} \text{out-degree}(V) = \sum_{v \in V} \text{in-degree}(V) = |E|
\]