VIII. NP-completeness
NP-Completeness – overview

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Tractable and intractable problems
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Tractable and intractable problems

- Problems that are solvable by polynomial-time algorithms are tractable
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- Problems that are solvable by \textit{polynomial-time} algorithms are \textit{tractable}.
- Problems that require \textit{superpolynomial time} are \textit{intractable}.

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Tractable and intractable problems

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- Problems that require superpolynomial time are intractable.

*Almost all the algorithms we have studied thus far have been polynomial-time algorithms on inputs of size \( n \), their worst-case running time is \( O(n^k) \) for some constant \( k \).*
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NP-complete (NPC) problems: an informal definition
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A large class of very diverse problems share the following properties:

1. We *only know* how to solve those problems in time much larger than polynomial, namely exponential time.
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NP-complete (NPC) problems: an informal definition

A large class of very diverse problems share the following properties:

1. We *only know* how to solve those problems in time much larger than polynomial, namely exponential time.

2. If we could *solve one NPC problem* in polynomial time, then there is a way to *solve every NPC problem* in polynomial time.
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Reasons to study NPC problems – practical
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- you can settle for approximating the solution, e.g., finding a nearly best solution rather than the optimum; or
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- you can use a known algorithm for an intractable problem, and accept that it will take a long long time to solve; or

- you can settle for approximating the solution, e.g., finding a nearly best solution rather than the optimum; or

- you can change your problem formulation so that it is solvable in polynomial time.
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- Indeed, this is one of the most famous problems in computer science:

\[ P \neq NP \]

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Whether NPC problems have polynomial solutions?
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- Indeed, this is one of the most famous problems in computer science: $P \overset{?}{=} NP$

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  Whether NPC problems have polynomial solutions?

- First posed in 1971
  http://www.claymath.org/millennium-problems
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P-vs-NP Examples

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Example 1.
- **Shortest path:** finding the shortest path from a single source in a directed graph.
- **Longest path:** finding the longest simple path between two vertices in a directed graph.

The first one is solvable in polynomial time (the Bellman-Ford algorithm), and the second is NPC, but the difference appears to be slight.
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P-vs-NP Examples

Example 2. ▶ Euler tour: given a connected, directed graph $G$, is there a cycle that visits each edge exactly once (although it is allowed to visit each vertex more than once)?
▶ Hamiltonian cycle: given a connected directed graph $G$, is there a simple cycle that visits each vertex exactly once?

The first one is solvable in polynomial time $^1$, and the second is NPC, but the difference appears to be slight.

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$^1$ Euler cycle of $G = (V, E)$ iff in-degree($v$) = out-degree($v$) for $\forall v \in V$. 

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P-vs-NP Examples

Example 3.

- **Minimum spanning tree (MST):** given a weighted graph and an integer $k$, is there a spanning tree whose total weight is $k$ or less?

- **Traveling salesperson problem (TSP):** given a weighted graph and an integer $k$, is there a cycle that visits all vertices exactly once whose total weight is $k$ or less?

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Remarks:

- To simplify discussion, we can consider only decision problems, rather than optimization problems.
- The optimization problems are at least as hard to solve as the related decision problems, we have not lost anything essential by doing so.
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Optimization-vs-Decision Examples

Example 1
Graph coloring: A coloring of a graph $G = (V, E)$ is a mapping $C: V \rightarrow S$ where $S$ is a finite set of "colors", such that $(u, v) \in E \Rightarrow C(u) \neq C(v)$.

- Optimization problem: given $G$, determine the smallest number of colors needed.
- Decision problem: given $G$ and a positive integer $k$, is there a coloring of $G$ using at most $k$ colors?
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Optimization-vs-Decision Examples

Example 2. Hamiltonian cycle: A Hamiltonian cycle is a cycle that passes through every vertex exactly once.

▶ decision problem: Does a given graph have a Hamiltonian cycle?
▶ optimization problem: Give a list of vertices of a Hamiltonian cycle.
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Optimization-vs-Decision Examples

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Optimization-vs-Decision Examples

Example 3. TSP (Traveling Salesperson Problem): given a weighted graph and an integer \( k \), is there a cycle that visits all vertices exactly once (Hamiltonian cycle) whose total weight is \( k \) or less?

- optimization problem: given a weighted graph, find a minimum Hamiltonian cycle.
- decision problem: given a weighted graph and an integer \( k \), is there a Hamiltonian cycle with total weight at most \( k \)?
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1. Introduction – recap

1. Tractable and intractable problems
   polynomial-boundness: $O(n^k)$

2. NP-complete problems – informal definition

3. P vs NP
   difference may appear “only slightly”

4. Optimization problems and decision problems