II.3 Frequency Response Analysis via Algebraic Substructuring

Frequency response calculation

- Frequency response

$$
\begin{aligned}
H(\omega) &= l^T(K - \omega^2 M + \mathtt{i}\omega D)^{-1} b \\
&= l^T \left[K - \omega^2 M + i\omega \underline{(\alpha K + \beta M)}\right]^{-1} b \\
&\downarrow \ \sigma(\text{shift}) \\
&= l^T(\gamma_1 K^\sigma + \gamma_2 M)^{-1} b
\end{aligned}
$$

- $H(\omega)$-projection onto the AS subspace span$\{L^{-1}S_m\}$:

$$H_m(\omega) = l_m^T(\gamma_1 K_m^\sigma + \gamma_2 M_m)^{-1} b_m = l^T p_m(\omega)$$

- *frequency response equation*

$$\underbrace{(\gamma_1 K_m^\sigma + \gamma_2 M_m)}\, p_m(\omega) \equiv G_m(\omega) p_m(\omega) = b_m$$

- Global modes $\Phi = (\phi) = \left[\, \Phi_l \ \ \Phi_n \ \ \Phi_r \,\right]$ of (K_m^σ, M_m)

- Decomposition $p_m(\omega) = p_n(\omega) + p_t(\omega)$,
  where $p_n(\omega) \in \text{span}\{\Phi_n\}$ and $p_t(\omega) \in \text{span}\{\begin{bmatrix} \Phi_l & \Phi_r \end{bmatrix}$.
  Note that the orthogonal condition: $p_n^T(\omega)(K_m^\sigma, M_m)p_t^T(\omega) = 0$.
- Phase 1: compute $p_n(\omega)$ directly
  Let $p_n(\omega) = \Phi_n \eta_n(\omega)$ for some $\eta_n(\omega)$. Then
  $$G_m(\omega)\left(\Phi_n \eta_n(\omega) + p_t(\omega)\right) = b_m$$
  Pre-multiplying by Φ_n^T and apply the orthogonal condition, we have
  $$\eta_n(\omega) = \left(\Phi_n^T G_m(\omega)\Phi_n\right)^{-1} \Phi_n^T b_m$$
  Hence
  $$\begin{aligned} p_n(\omega) &= \Phi_n \eta_n(\omega) \\ &= \Phi_n(\gamma_1 \Theta_n^\sigma + \gamma_2 I)^{-1}\Phi_n^T b_m. \end{aligned}$$
- Phase 2: compute $p_t(\omega)$ using frequency sweeping iteration
  Note that
  $$G_m(\omega)p_t(\omega) = b_m - G_m(\omega)p_n(\omega)$$

It is anticipated $p_t(\omega)$ varies slowly with frequency, we can use an iterative refinement.

$$p_t^\ell(\omega) = p_t^{\ell-1}(\omega) + \Delta p_t^\ell(\omega)$$

with initial response $p_t^0(\omega)$ by (linear) extrapolation from previous frequency point.

Then apply the Gerlerkin projection approximation to solve

$$G_m(\omega)\Delta p_t^\ell(\omega) = r_m^{\ell-1}(\omega) \equiv b_m - G_m(\omega)p_m^\ell(\omega)$$

We have the following so-called *frequency sweeping iteration*:

$$p_t^\ell(\omega) = p_t^{\ell-1}(\omega) + \frac{1}{\gamma_1}\left[(K_m^\sigma)^{-1} - \Phi_n(\Theta_n^\sigma)^{-1}\Phi_n^T\right] r_m^{\ell-1}(\omega)$$

- ASFRA package = ASEIG + FRA

  [Bennighof'98, ..., Ko& B.'08, Meerbergen and B.'09]

Convergence analysis of FS – cutoff values for Φ_n

- Truncated modal residual

$$\underbrace{\Phi_t^T r_m^\ell(\omega)} = -\frac{\gamma_2}{\gamma_1}\,\Theta_t^\sigma\;\underbrace{\Phi_t^T r_m^{\ell-1}(\omega)}\,.$$

- Contraction for the truncated mode $\Phi_t = (\phi_k)$:

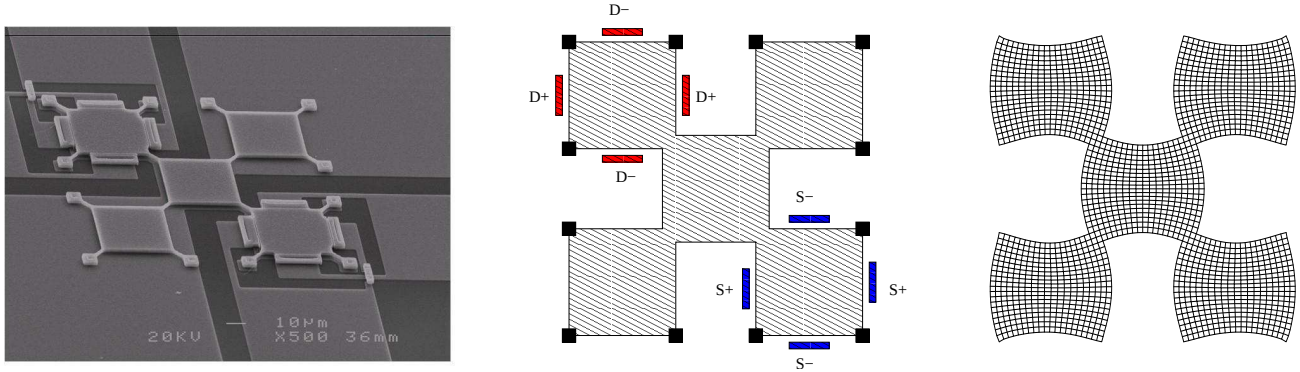
$$\left|\frac{\phi_k^T r_m^\ell(\omega)}{\phi_k^T r_m^{\ell-1}(\omega)}\right| \le \frac{d_{\max}}{|\theta_k^\sigma|} \le \xi < 1,$$

- Global cutoff values (to determine retained modes Φ_n) eigenpairs of the original matrix pair (K, M):

$$[\lambda_{\min},\ \lambda_{\max}] = \left[\sigma - \frac{d_{\max}}{\xi},\ \sigma + \frac{d_{\max}}{\xi}\right]$$

Case study: checkerboard filter

- FE simulation of a prototype checkerboard MEMS resonator for a high-frequency bandpass filter, *e.g.*, *the surface acoustic wave devices in a cell phone*

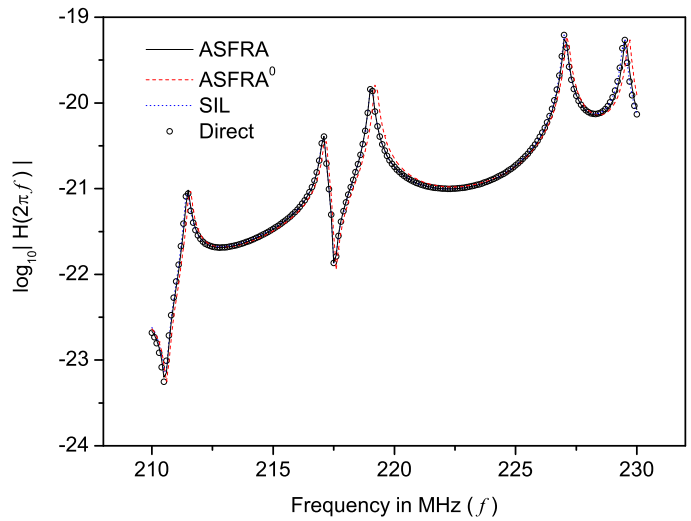


The SEM picture of a fabricated device and FE models, courtesy of Bindel *et al.*

- $N = 15258$, $[f_{\min}, f_{\max}] = [230, 250]\mathrm{MHz}$,

Case study: checkerboard filter – performance

- Accuracy of frequency responses:



- Performance

	Direct	SIL	ASFRA^0	ASFRA
m (AS subspace)	–	–	1635	307
n (retained modes)	–	242	231	37
Total FS iter	–	–	96	145
Elapsed time (sec.)	1612.6	86.23	208.15	26.72

Further reading

For further detail, see the following paper and references therein, available on the class website:

- J-H. Ko and Z. Bai, *High-frequency response analysis via algebraic substructuring* Inter J. Numer. Method in Engng., 76:295-313, 2008

A more recent paper is

- K. Meerbergen and Z. Bai, *The Lanczos method for parameterized symmetric linear systems with multiple right-hand sides*, submitted, 2009.