Basic Theory of Algebraic Eigenvalue Problems: *a quick review*

Essentials: definitions

Let $A \in \mathcal{C}^{n\times n}$.

1. A scalar λ is an **eigenvalue** of an $n \times n$ matrix A and a nonzero vector $x \in \mathcal{C}^n$ is a corresponding **(right) eigenvector** if

$$Ax = \lambda x.$$

   $\mathcal{L}_{A,\lambda} = \{x \, : \, Ax = \lambda x\}$ is an **eigenspace** of A.

2. A nonzero vector y such that

$$y^H A = \lambda y^H$$

   is a **left eigenvector**.

3. The set of all eigenvalues of A is called the **spectrum** of A.

4. $p_A(\lambda) = \det(\lambda I - A)$, a polynomial of degree n, is called the **characteristic polynomial** of A.

Essentials: properties

1. λ is A's eigenvalue $\Leftrightarrow \lambda I - A$ is singular $\Leftrightarrow \det(\lambda I - A) = 0$ $\Leftrightarrow p_A(\lambda) = 0$.
2. There is at least one eigenvector x associated with A's eigenvalue λ; in the other word, the dimension $\dim(\mathcal{L}_{A,\lambda}) \geq 1$.
3. $\mathcal{L}_{A,\lambda}$ is a linear subspace, i.e., it has the following two properties:
   (a) $x \in \mathcal{L}_{A,\lambda} \Rightarrow \alpha x \in \mathcal{L}_{A,\lambda}$ for all $\alpha \in \mathcal{C}$.
   (b) $x_1, x_2 \in \mathcal{L}_{A,\lambda} \Rightarrow x_1 + x_2 \in \mathcal{L}_{A,\lambda}$.
4. Suppose A is real. λ is A's eigenvalue $\Leftrightarrow$ conjugate $\bar{\lambda}$ is also A's eigenvalue.
5. A is singular $\Leftrightarrow 0$ is A's eigenvalue.
6. If A is upper (or lower) triangular, then its eigenvalues consist of its diagonal entries.

Invariant subspace

Definition. An **invariant subspace** of A is a subspace $\mathcal{V}$ of $\mathcal{R}^n$, with the property: $A\mathcal{V} \subseteq \mathcal{V}$

Examples

1. If x is an eigenvector, then span$\{x\}$ is an one-dimensional invariant subspace.
2. Let $x_1, x_2, \ldots, x_m$ be a set of independent eigenvectors associated with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then $\mathcal{X} =$ span$(\{x_1, x_2, \ldots, x_m\})$ is an invariant subspace.

Invariant subspace in matrix form Let $V = [v_1, v_2, \ldots, v_m]$ be any n-by-m matrix with linearly independent columns, and let $\mathcal{V} =$ span$\{V\}$, Then $\mathcal{V}$ is an invariant subspace *if and only if* there is an m-by-m matrix B such that

$$AV = VB.$$

Similarity transformation

Definitions.

- $n \times n$ matrices A and B are **similar** if there is an $n \times n$ non-singular matrix X such that $B = X^{-1}AX$.
- A is *unitarily similar* to B if X is unitary.

Properties: Suppose that A and B are similar: $B = X^{-1}AX$. Then

1. A and B have the same eigenvalues. In fact $p_A(\lambda) \equiv p_B(\lambda)$.
2. $Ax = \lambda x \Rightarrow B(X^{-1}x) = \lambda(X^{-1}x)$.
3. $Bw = \lambda w \Rightarrow A(Xw) = \lambda(Xw)$.

Basic task of eigenvalue computation

Let $X_1 \in \mathcal{C}^{n\times k}$, $k < n$, be a matrix with linearly independent columns that represents a nontrivial invariant subspace of A, i.e.,

$$AX_1 = X_1 B_{11}$$

for some $k \times k$ matrix B_{11}. Then let X_2 be a matrix such that $X = (X_1, X_2)$ is nonsingular, then it's easy to verify that

$$B = X^{-1}AX = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$

In other words, we can use X_1 to build a similarity transformation to block triangularlize A.

$\Longrightarrow$ **the basic task of eigenvalue computations is to find a nontrivial invariant subspace!**

Eigenvalue sensitivity

Let $Ax = \lambda x$, $y^H A = \lambda y^H$, and λ be the simple eigenvalue. Suppose A is perturbed to $\widetilde{A} \equiv A + \delta A$, and consequently λ is perturbed to $\widetilde{\lambda} \equiv \lambda + \delta\lambda$. If $\|\delta A\|_2 = \epsilon$ is sufficiently small, then

$$\delta\lambda = \frac{y^H(\delta A)x}{y^H x} + \mathcal{O}(\epsilon^2).$$

This implies

$$|\delta\lambda| \leq \underbrace{\frac{\|y\|_2\|x\|_2}{|y^H x|}}_{\text{cond}(\lambda)} \|\delta A\|_2 + \mathcal{O}(\epsilon^2).$$