SOLVING RATIONAL EIGENVALUE PROBLEMS VIA LINEARIZATION

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Abstract. Rational eigenvalue problem is an emerging class of nonlinear eigenvalue problems arising from a variety of physical applications. In this paper, we propose a linearization-based method to solve the rational eigenvalue problem. The proposed method converts the rational eigenvalue problem into a well-studied linear eigenvalue problem, and meanwhile, exploits and preserves the structure and properties of the original rational eigenvalue problem. For example, the low-rank property leads to a trimmed linearization. We show that solving a class of rational eigenvalue problems is just as convenient and efficient as solving linear eigenvalue problems of slightly larger sizes.

Key words. Rational eigenvalue problem, linearization

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1. Introduction. In recent years, there are a great deal of interest to study the rational eigenvalue problem (REP)

$$R(\lambda)x = 0, \tag{1.1}$$

where $R(\lambda)$ is an $n \times n$ matrix rational function of the form

$$R(\lambda) = P(\lambda) - \sum_{i=1}^{k} \frac{s_i(\lambda)}{q_i(\lambda)} E_i, \tag{1.2}$$

$P(\lambda)$ is an $n \times n$ matrix polynomial in λ of degree d, $s_i(\lambda)$ and $q_i(\lambda)$ are scalar polynomials of degrees n_i and d_i, respectively, and E_i are $n \times n$ constant matrices. The REP (1.2) arises from optimization of acoustic emissions of high speed trains [13], free vibration of plates with elastically attached masses [20], vibration of fluid-solid structure [21], free vibrations of a structure with a viscoelastic constitutive relation describing the behavior of a material [16] and electronic structure calculations of quantum dots [23, 10].

A brute-force approach to solve the REP (1.1) is to multiply equation (1.1) by the scalar polynomial $\prod_{i=1}^{k} q_i(\lambda)$ to turn it into a polynomial eigenvalue problem (PEP) of the degree $d_* = d + d_1 + \cdots + d_k$. Subsequently, the PEP is converted into a linear eigenvalue problem (LEP) by the process known as linearization. This approach is noted in [16] and employed in [10, 9] for electronic structure calculations of quantum dots. The recent study of the linearization techniques of the PEP can be found in [13, 14, 7, 8] and references therein. This is a practical approach only if the number k of the rational terms and the degree d_i of the polynomials $q_i(\lambda)$ are small, say $k = d_1 = 1$. However, when k and/or d_i are large, it leads to a PEP of much higher degree and becomes impractical for large-scale problems. Furthermore, the possible low-rank property of the matrices E_i is lost in the linearized eigenvalue problem.

An alternative approach is to treat the REP (1.1) as a general nonlinear eigenvalue problem (NEP), and solve it by a nonlinear eigensolver, such as Picard iteration (self-consistent iteration), Newton's method, nonlinear Rayleigh quotient method, nonlinear Jacobi-Davidson method, and nonlinear Arnoldi method [16, 18, 20, 22]. This approach limits the exploitation of the underlying rich structure and property of the REP (1.1), and is challenging in convergence analysis and validation of computed eigenvalues.

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In this paper, we propose a linearization-based approach to solve the REP (1.1). Similarly to the linearization of the PEP, the new approach converts the REP (1.1) into a linear eigenvalue problem (LEP), exploits and preserves the structure and property of the REP (1.1) as much as possible. It has a number of advantages. For example, the low-rank property of matrices E_i as frequently encountered in applications leads to a trimmed linearization, namely, only small increase of the size comparing to the size of the original REP (1.1). The symmetry of the REP can also be preserved in the LEP. We show that under mild assumptions, the problem of solving a class of the REPs is just as convenient and efficient as the problem of solving an LEP of slightly larger size.

The rest of this paper is organized as follows. In section 2, we formalize the definition of the REP (1.1) and the assumptions we will use throughout. In section 3, we present a linearization scheme. In section 4, we show how to use the proposed linearization scheme to a number of REPs from different applications. Numerical examples are given in section 5.

2. Settings. We assume throughout that the matrix rational function $R(\lambda)$ is *regular*, that is, $\det(R(\lambda)) \not\equiv 0$. The roots of $q_i(\lambda)$ are the *poles* of $R(\lambda)$. $R(\lambda)$ is not defined on these poles. A scalar λ such that $\det(R(\lambda)) = 0$ is referred to as an *eigenvalue*, and the corresponding nonzero vector x satisfying the equation (1.1) is called an *eigenvector*. The pair (λ, x) is referred to as an eigenpair.

Let us denote the matrix polynomial $P(\lambda)$ of degree d in λ as

$$P(\lambda) = \lambda^d A_d + \lambda^{d-1} A_{d-1} + \cdots + \lambda A_1 + A_0, \tag{2.1}$$

where A_i are $n \times n$ constant matrices. We assume throughout that the leading coefficient matrix A_d is *nonsingular*, which is equivalent to the assumption of a monic matrix polynomial ($A_d = I$) in the study of matrix polynomial [5]. The treatment in the presence of singular A_d is beyond the scope of this paper.

We assume that $s_i(\lambda)$ and $q_i(\lambda)$ are *coprime*, that is, having no common factors. Furthermore, the rational functions $\frac{s_i(\lambda)}{q_i(\lambda)}$ are *proper*, that is, $s_i(\lambda)$ having smaller degree than $q_i(\lambda)$. Otherwise by the polynomial long division, an improper rational function can be written as the sum of a polynomial and a proper rational function:

$$\frac{s_i(\lambda)}{q_i(\lambda)} = p_i(\lambda) + \frac{\widehat{s}_i(\lambda)}{q_i(\lambda)}$$

with $\widehat{s}_i(\lambda)$ having smaller degree than $q_i(\lambda)$. Subsequently, the term $p_i(\lambda)E_i$ can be absorbed into the matrix polynomial term $P(\lambda)$:

$$P(\lambda) := P(\lambda) + p_i(\lambda)E_i.$$

If it is necessary, we assume that the leading coefficient matrix of the updated matrix polynomial is still nonsingular.

Since $s_i(\lambda)$ and $q_i(\lambda)$ are coprime, the proper rational function $\frac{s_i(\lambda)}{q_i(\lambda)}$ can be represented as

$$\frac{s_i(\lambda)}{q_i(\lambda)} = a_i^T (C_i - \lambda D_i)^{-1} b_i, \tag{2.2}$$

for some matrices $C_i, D_i \in \mathbb{R}^{d_i \times d_i}$, and vectors $a_i, b_i \in \mathbb{R}^{d_i \times 1}$. Moreover, D_i is nonsingular. The process of constructing the quadruple (C_i, D_i, a_i, b_i) satisfying (2.2) is called a *minimal realization* in the theory of control system, see for example [1, pp.91–98] and [19].

Finally, we assume that coefficient matrices E_i have the rank-revealing decompositions

$$E_i = L_i U_i^T, \tag{2.3}$$

where $L_i, U_i \in \mathbb{R}^{n \times r_i}$ are of full column rank r_i. In section 4, we will see that the decompositions (2.3) are often immediately available in the practical REPs. The rank of E_i is typically much smaller than the size n, that is, $r_i \ll n$. The algorithms for computing such sparse rank-revealing decompositions can be found in [17].

3. Linearization. Under the assumptions discussed in the previous section, let us consider a linearization method for solving the REP (1.1). By the realizations (2.2) of the rational functions $s_i(\lambda)/q_i(\lambda)$ and the factorizations (2.3) of the coefficient matrices E_i, the rational terms of the matrix rational function $R(\lambda)$ can be rewritten as the following

$$\begin{aligned}\sum_{i=1}^{k}\frac{s_i(\lambda)}{q_i(\lambda)}E_i &= \sum_{i=1}^{k} a_i^T(C_i-\lambda D_i)^{-1}b_iL_iU_i^T\\ &= \sum_{i=1}^{k} L_i\left[a_i^T(C_i-\lambda D_i)^{-1}b_i\cdot I_{r_i}\right]U_i^T\\ &= \sum_{i=1}^{k} L_i(I_{r_i}\otimes a_i)^T(I_{r_i}\otimes C_i-\lambda\, I_{r_i}\otimes D_i)^{-1}(I_{r_i}\otimes b_i)U_i^T,\end{aligned}$$

where $\otimes$ is the Kronecker product. Define

$$\begin{aligned}C &= \mathrm{diag}(I_{r_1}\otimes C_1,\, I_{r_2}\otimes C_2,\, \ldots,\, I_{r_k}\otimes C_k),\\ D &= \mathrm{diag}(I_{r_1}\otimes D_1,\, I_{r_2}\otimes D_2,\, \ldots,\, I_{r_k}\otimes D_k),\\ L &= \left[\begin{array}{cccc} L_1(I_{r_1}\otimes a_1)^T & L_2(I_{r_2}\otimes a_2)^T & \cdots & L_k(I_{r_k}\otimes a_k)^T\end{array}\right],\\ U &= \left[\begin{array}{cccc} U_1(I_{r_1}\otimes b_1)^T & U_2(I_{r_2}\otimes b_2)^T & \cdots & U_k(I_{r_k}\otimes b_k)^T\end{array}\right],\end{aligned}$$

where the size of C and D is $m\times m$, the size of L and U is $n\times m$, and $m=r_1d_1+r_2d_2+\cdots+r_kd_k$. Then the rational terms of $R(\lambda)$ can be compactly represented in a realization form

$$\sum_{i=1}^{k}\frac{s_i(\lambda)}{q_i(\lambda)}E_i = L(C-\lambda D)^{-1}U^T. \tag{3.1}$$

We note that the matrix D is nonsingular since the matrices D_i in (2.2) are nonsingular. The eigenvalues of the matrix pencil $C-\lambda D$ are the poles of $R(\lambda)$.

Using the representation (3.1), the REP (1.1) can be equivalently written in the following compact form

$$\left[P(\lambda)-L(C-\lambda D)^{-1}U^T\right]x=0, \tag{3.2}$$

and the matrix rational function $R(\lambda)$ is written as

$$R(\lambda)=P(\lambda)-L(C-\lambda D)^{-1}U^T. \tag{3.3}$$

If $P(\lambda)$ is linear and is denoted as $P(\lambda)=A-\lambda B$, then the REP (3.2) is of the form

$$\left[A-\lambda B-L(C-\lambda D)^{-1}U^T\right]x=0. \tag{3.4}$$

By introducing the auxiliary vector

$$y=-(C-\lambda D)^{-1}U^Tx,$$

the equation (3.2) can be written as the following LEP:

$$(\mathcal{A}-\lambda\mathcal{B})z=0, \tag{3.5}$$

where

$$\mathcal{A}=\left[\begin{array}{cc} A & L\\ U^T & C\end{array}\right],\quad \mathcal{B}=\left[\begin{array}{cc} B & \\ & D\end{array}\right],\quad z=\left[\begin{array}{c} x\\ y\end{array}\right].$$

In general, if the matrix polynomial $P(\lambda)$ is of the form (2.1), we can first write the REP (3.2) as a semi-PEP of the form

$$\left(\lambda^d A_d + \lambda^{d-1} A_{d-1} + \cdots + \lambda A_1 + \widetilde{A}_0(\lambda)\right) x = 0, \tag{3.6}$$

where $\widetilde{A}_0(\lambda) \triangleq A_0 - L(C - \lambda D)^{-1} U^T$. Then by symbolically applying the well-known (first) companion form linearization to the semi-PEP (3.6), we have

$$\left(\begin{bmatrix} A_{d-1} & A_{d-2} & \cdots & \widetilde{A}_0(\lambda) \\ -I & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & -I & 0 \end{bmatrix} - \lambda \begin{bmatrix} A_d & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix} \right) \begin{bmatrix} \lambda^{d-1} x \\ \lambda^{d-2} x \\ \vdots \\ x \end{bmatrix} = 0, \tag{3.7}$$

which can be equivalently written as

$$\left(\begin{bmatrix} A_{d-1} & A_{d-2} & \cdots & A_0 \\ -I & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & -I & 0 \end{bmatrix} - \lambda \begin{bmatrix} A_d & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix} - \begin{bmatrix} L \\ 0 \\ \vdots \\ 0 \end{bmatrix} (C - \lambda D)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ U \end{bmatrix}^T \right) \begin{bmatrix} \lambda^{d-1} x \\ \lambda^{d-2} x \\ \vdots \\ x \end{bmatrix} = 0.$$

The above equation is of the same form as (3.4). Therefore, by introducing the variable

$$y = -(C - \lambda D)^{-1} \begin{bmatrix} 0 & 0 & \cdots & U^T \end{bmatrix} \begin{bmatrix} \lambda^{d-1} x \\ \lambda^{d-2} x \\ \vdots \\ x \end{bmatrix} = -(C - \lambda D)^{-1} U^T x,$$

we derive the following linearization of the REP (1.1):

$$(\mathcal{A} - \lambda \mathcal{B}) z = 0, \tag{3.8}$$

where

$$\mathcal{A} = \left[\begin{array}{cccc|c} A_{d-1} & A_{d-2} & \cdots & A_0 & L \\ -I & 0 & \cdots & 0 & \\ & \ddots & \ddots & \vdots & \\ & & -I & 0 & \\ \hline & & & U^T & C \end{array}\right], \quad \mathcal{B} = -\left[\begin{array}{cccc|c} A_d & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \\ \hline & & & & -D \end{array}\right], \quad z = \left[\begin{array}{c} \lambda^{d-1} x \\ \lambda^{d-2} x \\ \vdots \\ x \\ \hline y \end{array}\right].$$

The size of matrices $\mathcal{A}$ and $\mathcal{B}$ is $nd + m$, where $m = r_1 d_1 + r_2 d_2 + \cdots + r_k d_k$. In the case that all the coefficient matrices E_i are of full rank, i.e., $r_i = n$, the LEP (3.8) is of the size nd_*, where $d_* = d + d_1 + \cdots + d_k$. This is the same size as the one derived by the brute-force approach. However, it is typical that $r_i \ll n$ in practice, then $nd + m \ll nd_*$. The LEP (3.8) is a trimmed linearization of the REP (1.1). This will be illustrated by four REPs from applications in section 4.

Note that under the assumption of nonsingularity of the matrix A_d, the matrix $\mathcal{B}$ is nonsingular. Therefore all eigenvalues of the LEP (3.8) are finite. There is no infinite eigenvalue.

The following theorem shows the connection between eigenvalues of the REP (1.1) and the LEP (3.8).

THEOREM 3.1. *(a) If λ is an eigenvalue of the REP (1.1), then it is an eigenvalue of the LEP (3.8).*

(b) If λ is an eigenvalue of the LEP (3.8) and is not a pole of $R(\lambda)$, then it is an eigenvalue of the REP (1.1).

Proof. Define dn-by-dn matrices

$$V_L(\lambda) = \begin{bmatrix} I & -\lambda A_d - A_{d-1} & -A_{d-2} & \cdots & -A_1 \\ & I & -\lambda I & & \\ & & \ddots & \ddots & \\ & & & \ddots & -\lambda I \\ & & & & I \end{bmatrix},$$

$$V_R(\lambda) = \begin{bmatrix} \lambda^{d-1} I & -I & & \\ \vdots & & \ddots & \\ \lambda I & & & -I \\ I & & & \end{bmatrix}.$$

We have $\det(V_L(\lambda)) = \det(V_R(\lambda)) = 1$ and

$$\begin{bmatrix} \lambda A_d + A_{d-1} & A_{d-2} & \cdots & A_0 \\ -I & \lambda I & & \\ & \ddots & \ddots & \\ & & -I & \lambda I \end{bmatrix} V_R(\lambda) = V_L(\lambda) \begin{bmatrix} P(\lambda) & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \det(\mathcal{A} - \lambda\mathcal{B}) &= \det\left((\mathcal{A} - \lambda\mathcal{B}) \left[\begin{array}{c|c} V_R(\lambda) & \\ \hline & I_m \end{array}\right] \right) \\ &= \det\left(\left[\begin{array}{cccc|c} \lambda A_d + A_{d-1} & A_{d-2} & \cdots & A_0 & L \\ -I & \lambda I & & & \\ & \ddots & \ddots & & \\ & & -I & \lambda I & \\ \hline & & & U^T & C - \lambda D \end{array}\right] \left[\begin{array}{c|c} V_R(\lambda) & \\ \hline & I_m \end{array}\right] \right) \\ &= \det\left(\left[\begin{array}{c|c} V_L(\lambda) & \\ \hline & I_m \end{array}\right] \left[\begin{array}{ccc|c} P(\lambda) & & & L \\ & I & & \\ & & \ddots & \\ & & I & \\ \hline U^T & & & C - \lambda D \end{array}\right] \right) \\ &= \det\left(\left[\begin{array}{ccc|c} P(\lambda) & & & L \\ & I & & \\ & & \ddots & \\ & & I & \\ \hline U^T & & & C - \lambda D \end{array}\right] \right), \end{aligned} \tag{3.9}$$

where for the third equality, we use the identities

$$\begin{bmatrix} 0 & \cdots & 0 & U^T \end{bmatrix} V_R(\lambda) = \begin{bmatrix} U^T & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} L \\ 0 \\ \vdots \\ 0 \end{bmatrix} = V_L(\lambda) \begin{bmatrix} L \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By exploiting the block structure of the matrix in the determinant of the right-hand-side of the equation (3.9), we derive that

$$\det(\mathcal{A} - \lambda\mathcal{B}) = \det\left(\begin{bmatrix} P(\lambda) & L \\ U^T & C - \lambda D \end{bmatrix}\right). \tag{3.10}$$

If λ is an eigenvalue of the REP (1.1), then it is not a pole of the REP. It implies that λ is not an eigenvalue of $C - \lambda D$ and $C - \lambda D$ is nonsingular. Therefore, we have the block factorization

$$\begin{bmatrix} P(\lambda) & L \\ U^T & C - \lambda D \end{bmatrix} = \begin{bmatrix} I & L(C - \lambda D)^{-1} \\ & I \end{bmatrix} \begin{bmatrix} R(\lambda) & \\ U^T & C - \lambda D \end{bmatrix},$$

where $R(\lambda)$ is defined by (3.3). The proof of part (a) immediately follows the identity

$$\det(\mathcal{A} - \lambda\mathcal{B}) = \det(R(\lambda)) \cdot \det(C - \lambda D). \tag{3.11}$$

For the part (b), if λ is an eigenvalue of the pencil $\mathcal{A} - \lambda\mathcal{B}$ and is not the pole of $R(\lambda)$, then the matrix $C - \lambda D$ is nonsingular. By the identity (3.11), λ is an eigenvalue of the REP (1.1). □

We note that the condition that λ is not a pole of the $R(\lambda)$ in Theorem 3.1(b) is necessary. Consider the following example:

$$\left(\lambda I_2 - \frac{1}{\lambda} e_2 e_2^T\right) x = 0, \tag{3.12}$$

where I_2 is a 2 by 2 identity matrix, and e_2 is the second column of I_2. Since $\det(R(\lambda)) = \lambda(\lambda - \frac{1}{\lambda})$, the REP (3.12) has two eigenvalues 1 and -1. $\lambda = 0$ is a pole. Let $y = \lambda^{-1} e_2^T x$, then the corresponding LEP is given by

$$(\mathcal{A} - \lambda\mathcal{B})z = \left(\left[\begin{array}{c|c} 0 & e_2 \\ \hline e_2^T & 0 \end{array}\right] - \lambda \left[\begin{array}{c|c} I_2 & \\ \hline & 1 \end{array}\right]\right) \left[\begin{array}{c} x \\ \hline y \end{array}\right] = 0.$$

It has three eigenvalues, $-1, 0$ and 1. But $\lambda = 0$ is not an eigenvalue of the REP (3.12).

Two additional remarks are in order. First, the realization of a rational function can be represented in different forms. For example, the realization of $\frac{1}{(\sigma-\lambda)^2}$ could be given by

$$\frac{1}{(\sigma - \lambda)^2} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} \sigma & -1 \\ 0 & \sigma \end{bmatrix} - \lambda I\right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or in a symmetric form

$$\frac{1}{(\sigma - \lambda)^2} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} & \sigma \\ \sigma & -1 \end{bmatrix} - \lambda \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The study of realization can be found in [1, pp.91–98] and references therein.

Second, there are many different ways of linearization for the matrix polynomials, including recent work [7, 8, 13, 14]. Many of these linearizations can be easily integrated into the proposed

linearization of the REP. For example, if the original REP (1.1) is symmetric, namely A_i are symmetric, matrices C and D are symmetric in the realization and $L = U$, then we can use a symmetric linearization proposed in [7]. Specifically, let us consider a symmetric matrix polynomial of degree $d = 3$, then a symmetric linearization is given by

$$\left(\begin{bmatrix} & -A_3 & & \\ -A_3 & -A_2 & & \\ & & A_0 & U \\ & & U^T & C \end{bmatrix} + \lambda \begin{bmatrix} & & A_3 & \\ & A_3 & A_2 & \\ A_3 & A_2 & A_1 & \\ & & & -D \end{bmatrix} \right) \begin{bmatrix} \lambda^2 x \\ \lambda x \\ x \\ y \end{bmatrix} = 0, \tag{3.13}$$

where $y = -(C - \lambda D)^{-1} U^T x$.

4. Applications. In this section, we apply the proposed linearization in section 3 to four REPs from applications.

4.1. Loaded elastic string. We consider the following rational eigenvalue problem arising from the finite element discretization of a boundary value problem describing the eigenvibration of a string with a load of mass attached by an elastic spring:

$$\left(A - \lambda B + \frac{\lambda}{\lambda - \sigma} E \right) x = 0, \tag{4.1}$$

where A and B are tridiagonal and symmetric positive definite, and $E = e_n e_n^T$, e_n is the last column of the identity matrix. σ is a parameter [2, 20].

By the linearization proposed in section 3, the first step is to write the rational function $\lambda/(\lambda - \sigma)$ in a proper form. It results that the REP (4.1) becomes

$$\left(A + e_n e_n^T - \lambda B + \frac{\sigma}{\lambda - \sigma} e_n e_n^T \right) x = 0. \tag{4.2}$$

Then it can be easily written in the realization form (3.4):

$$\left[A + e_n e_n^T - \lambda B - e_n \left(1 - \frac{\lambda}{\sigma} \right)^{-1} e_n^T \right] x = 0. \tag{4.3}$$

By defining the auxiliary vector

$$y = - \left(1 - \frac{\lambda}{\sigma} \right)^{-1} e_n^T x,$$

we have the linear eigenvalue problem

$$(\mathcal{A} - \lambda \mathcal{B}) z = 0, \tag{4.4}$$

where

$$\mathcal{A} = \begin{bmatrix} A + e_n e_n^T & e_n \\ e_n^T & 1 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B & \\ & 1/\sigma \end{bmatrix}, \quad z = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The $(n+1) \times (n+1)$ matrices $\mathcal{A}$ and $\mathcal{B}$ have the same structure and property as the coefficient matrices A and B in the original REP (4.1). They are tridiagonal and symmetric positive definite.

An alternative way to solve the REP (4.1) is to first transform it to a quadratic eigenvalue problem (QEP) by multiplying the linear factor $\lambda - \sigma$ on the equation (4.1):

$$Q(\lambda) x = \left[\lambda^2 B - \lambda (A + \sigma B + e_n e_n^T) + \sigma A \right] x = 0, \tag{4.5}$$

and then convert the QEP (4.5) into an equivalent linear eigenvalue problem by the linearization process. A symmetric linearization is

$$\left(\begin{bmatrix} -(A+\sigma B+e_ne_n^T) & \sigma A \\ \sigma A & \end{bmatrix}+\lambda\begin{bmatrix} B & \\ & -\sigma A \end{bmatrix}\right)\begin{bmatrix} \lambda x \\ x \end{bmatrix}=0. \tag{4.6}$$

This is a generalized symmetric indefinite eigenvalue problem. If $Q(a)<0$ for some scalar a, then by letting $\lambda=\mu+a$, the QEP (4.5) becomes

$$\left[\mu^2B-\mu(A+(\sigma-2a)B+e_ne_n^T)+Q(a)\right]x=0.$$

Consequently, it can be linearized to a generalized symmetric definite eigenvalue problem:

$$\left(\begin{bmatrix} -\left(A+(\sigma-2a)B+e_ne_n^T\right) & Q(a) \\ Q(a) & 0 \end{bmatrix}+\mu\begin{bmatrix} B & \\ & -Q(a) \end{bmatrix}\right)\begin{bmatrix} \mu x \\ x \end{bmatrix}=0. \tag{4.7}$$

A practical issue is on the existence of the shift a and how to find it numerically, see recent work [6].

We note that the size of the LEP (4.6) and (4.7) is $2n$. On the other hand, the size of the LEP (4.4) by the new linearization process is only $n+1$.

4.2. Quadratic eigenvalue problem with low-rank stiffness matrix. Consider the QEP:

$$(\lambda^2M+\lambda D+K)x=0. \tag{4.8}$$

If K is singular, then zero is an eigenvalue since $Kx=0$ for some nonzero vector x. Let us consider how to compute nonzero eigenvalues of the QEP (4.8) by exploiting the fact that K is singular and is of low rank. Let

$$K=LU^T$$

be the full-rank decomposition, where $L,U\in\mathbb{R}^{n\times r}$, r is the rank of K. By the linearization discussed in section 3, the QEP (4.8) can be written in the REP form (1.1)

$$\left(\lambda M+D+\frac{1}{\lambda}LU^T\right)x=0.$$

In the compact form (3.2), it becomes

$$\left[\lambda M+D-L(0-\lambda I)^{-1}U^T\right]x=0.$$

Note that the polynomial term $P(\lambda)=\lambda M+D$ is linear. Hence, by introducing the auxiliary vector

$$y=-(0-\lambda I)^{-1}U^Tx=\lambda^{-1}U^Tx,$$

we have the LEP

$$\left(\lambda\begin{bmatrix} M & \\ & I \end{bmatrix}+\begin{bmatrix} D & L \\ -U^T & 0 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}=0. \tag{4.9}$$

If $L=K,U=I$, then the LEP (4.9) is a linearization of the QEP in the first companion form. If $L=-I,U=-K^T$, then the LEP (4.9) is a linearization in the second companion form [11]. If K has rank $r<n$, the linearization (4.9) is not a linearization of the QEP (4.8) under the standard definition of linearization of the PEPs [5]. However, by Theorem 3.1, we can conclude that the nonzero eigenvalues of QEP (4.8) and LEP (4.9) are the same. The LEP (4.9) is a trimmed linearization of the QEP. The order of the LEP (4.9) $n+r$ could be significantly smaller than the size $2n$ of the linearization in the companion forms.

If M is singular or more general, the leading coefficient matrix A_d is singular in the matrix polynomial, the PEP $P(\lambda)=0$ has infinite eigenvalues and/or corresponding singular structure. It is discussed in [3] that how to exploit the structure of the zero blocks in the coefficient matrices to obtain a trimmed linearization by (partially) deflating those infinite eigenvalues and/or singular structures. In the linearization (4.9), by using a full-rank factorization of the stiffness matrix K, we derived a trimmed linearization such that those zero eigenvalues are explicitly deflated.

4.3. Vibration of a fluid-solid structure. Let us consider an REP arising from the simulation of mechanical vibrations of fluid-solid structures [15, 16, 21]. It is of the form

$$\left(A - \lambda B + \sum_{i=1}^{k} \frac{\lambda}{\lambda - \sigma_i} E_i\right) x = 0, \tag{4.10}$$

where the poles $\sigma_i, i = 1, 2, \ldots, k$ are positive, matrices A and B are symmetric positive definite, and

$$E_i = C_i C_i^T,$$

$C_i \in \mathbb{R}^{n \times r_i}$ has rank r_i for $i = 1, 2, \ldots, k$.

By the linearization proposed in section 3, we first write the rational terms of (4.10) in the proper form

$$\left(A + \sum_{i=1}^{k} C_i C_i^T - \lambda B - \sum_{i=1}^{k} \frac{\sigma_i}{\sigma_i - \lambda} C_i C_i^T\right) x = 0. \tag{4.11}$$

Let

$$C = \begin{bmatrix} C_1 & C_2 & \cdots & C_k \end{bmatrix}, \quad \Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, \ldots, \sigma_k I_{r_k}),$$

where I_{r_i} is the r_i-by-r_i identity. Then the equation (4.11) can be written as

$$\left[A + CC^T - \lambda B - C(I - \lambda \Sigma^{-1})^{-1} C^T\right] x = 0.$$

By introducing the variable $y = -(I - \lambda\Sigma^{-1})^{-1} C^T x$, we have the following LEP:

$$(\mathcal{A} - \lambda \mathcal{B}) \begin{bmatrix} x \\ y \end{bmatrix} = 0, \tag{4.12}$$

where

$$\mathcal{A} = \begin{bmatrix} A + CC^T & C \\ C^T & I \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B & \\ & \Sigma^{-1} \end{bmatrix}$$

are of the size $n + \sum_{i=1}^{k} r_i$. Note that the matrix $\mathcal{A}$ is symmetric, and $\mathcal{B}$ is symmetric positive definite. The LEP (4.12) is a generalized symmetric definite eigenvalue problem, which can be essentially solved by a symmetric eigensolver, such as implicitly restarted Lanczos algorithm [12] or the thick-restart Lanczos method [24].

Mazurenko and Voss [15] addressed the question to determine the number of eigenvalues of the REP (4.10) in a given interval (α, β) by using the fact that the eigenvalues of (4.10) can be characterized as minimax values of a Rayleigh functional [15, p.610]. By using the linearization (4.12), the question can be answered through computing the inertias of the symmetric matrix pencil $\mathcal{A} - \lambda\mathcal{B}$. First, we have the following proposition.

PROPOSITION 4.1. *If A and B are positive definite and the poles $\sigma_i > 0$, then all eigenvalues of the REP (4.10) are real and positive.*

Proof. Since the matrix pencil $\mathcal{A} - \lambda\mathcal{B}$ in (4.12) is a symmetric definite pencil, all eigenvalues are real. By the fact that $\mathcal{A}$ is semi-positive definite, all eigenvalues are nonnegative. Therefore, we just need to show that zero is not an eigenvalue of the pencil $\mathcal{A} - \lambda\mathcal{B}$. By contradiction, let $z = \begin{bmatrix} x^T & y^T \end{bmatrix}^T \neq 0$ be an eigenvector corresponding to the zero eigenvalue. Then by $(\mathcal{A} - 0 \cdot \mathcal{B})z = \mathcal{A}z = 0$, we have

$$\begin{aligned} (A + CC^T)x + Cy &= 0, \\ C^T x + y &= 0. \end{aligned}$$

Note that A is positive definite, we have $x = 0$ and $y = 0$. This is a contradiction. Therefore all eigenvalues of the LEP (4.12) are positive. By Theorem 3.1, we conclude that all eigenvalues of the REP (4.10) are positive. □

Based on Theorem 3.1 and Proposition 4.1, we conclude that the number of eigenvalues of the REP (4.10) in the interval (α, β) is given by

$$\kappa = \ell - \ell_0, \tag{4.13}$$

where ℓ is the number of eigenvalues of $\mathcal{A} - \lambda\mathcal{B}$ in the interval (α, β), $\ell_0 = \sum_{\sigma_i \in (\alpha,\beta)} \ell_i$, and ℓ_i is the number of zero eigenvalues of $\mathcal{A} - \sigma_i\mathcal{B}$. The quantities ℓ and ℓ_0 can be computed using the Sylvester's law of inertia for the real symmetric matrices $\mathcal{A} - \tau\mathcal{B}$ for $\tau = \alpha, \beta$ and poles $\sigma_i \in (\alpha, \beta)$.

4.4. Damped vibration of a structure. This is an REP arising from the free vibrations of a structure if one uses a viscoelastic constitutive relation to describe the behavior of a material [16]. The REP is of the form

$$\left(\lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i\lambda} \Delta G_i\right) x = 0, \tag{4.14}$$

where the mass and stiffness matrices M and K are symmetric positive definite, b_j are relaxation parameters over the k regions, ΔG_j is an assemblage of element stiffness matrices over the region with the distinct relaxation parameters.

We consider the case where $\Delta G_i = L_i L_i^T$ and $L_i \in \mathbb{R}^{n \times r_i}$. By defining

$$L = [L_1, L_2, \ldots, L_k], \quad D = \mathrm{diag}(b_1 I_{r_1}, b_2 I_{r_2}, \ldots, b_k I_{r_k}),$$

the REP (4.14) can be written in the form (3.2):

$$(\lambda^2 M + K - L(I + \lambda D)^{-1} L^T)x = 0.$$

By linearizing the second-order matrix polynomial term $\lambda^2 M + K$ in a symmetric form, we derive the following symmetric LEP:

$$\left(\begin{bmatrix} -M & & \\ & K & L \\ & L^T & I \end{bmatrix} + \lambda \begin{bmatrix} & M & \\ M & & \\ & & D \end{bmatrix}\right) \begin{bmatrix} \lambda x \\ x \\ y \end{bmatrix} = 0,$$

where the auxiliary vector $y = -(I + \lambda D)^{-1} L^T x$. The size of the LEP is $2n + r_1 + r_2 + \cdots + r_k$.

5. Numerical examples. In this section, we present two numerical examples to show computational efficiency of the proposed linearization process of the REP (1.1) in sections 3 and 4. We do not compare the proposed approach with a general-purpose nonlinear eigensolver, such as Newton's method, nonlinear Arnoldi method[22] or preconditioned iterative methods [20]. Instead, we compare the extra cost of solving the REP (1.1) over the problem of solving the PEP $P(\lambda)x = 0$ without the rational terms in (1.1). All numerical experiments were run in MATLAB 7.0.1 on a Pentium IV PC with 2.6GHz CPU and 1GB of core memory.

Example 1. We present numerical results for the REP (4.1) arising from vibration analysis of a loaded elastic string discussed in section 4.1. This REP is included in the collection of nonlinear eigenvalue problems (NLEVP) [2]. If the NLEVP is included in MATLAB , then the matrices A, B and E can be generated by calling

```
coeffs = nlevp('loaded_string',n);
A = coeffs{1};  B = coeffs{2}; E = coeffs{3};
```

where n is the size of the REP. As in [20], the pole σ is set to be 1. The interested eigenvalues are few smallest ones in the interval $\lambda \in (1, +\infty)$.

The following table records the 10 computed smallest eigenvalues and the corresponding residual norms of the trimmed LEP (4.4) with the size $n = 100$ by MATLAB function `eig`:

i	$\widehat{\lambda}_i$	residual norm
1	0.457318488953671	$5.58e-013$
2	4.48217654587198	$5.96e-013$
3	24.2235731125539	$6.69e-013$
4	63.7238211419405	$9.40e-013$
5	123.031221067605	$8.63e-013$
6	202.200899143561	$9.56e-013$
7	301.310162794155	$1.09e-012$
8	420.456563106511	$1.01e-012$
9	559.757586307048	$7.12e-013$
10	719.350660116386	$9.15e-013$

The residual norm $\|R(\widehat{\lambda})\widehat{x}\|_2/\|\widehat{x}\|_2$ is used to measure the precision of a computed eigenpair $(\widehat{\lambda},\widehat{x})$ of REP (1.1), the same as in [16, Algorithm 5].

We note that the first eigenvalue $\widehat{\lambda}_1 < 1$, which is not of practical interest according to [20]. Eigenvalues $\widehat{\lambda}_2$ to $\widehat{\lambda}_6$ match all significant digits of the computed eigenvalues by a preconditioned iterative method reported in [20].

The following table reports the CPU elapsed time for solving the trimmed LEP (4.4) for different sizes n by using MATLAB dense eigensolver `eig`. For comparison, we also report the CPU elapsed time of solving the QEP (4.5) by using MATLAB function `polyeig`, and the symmetric LEP (4.7) of size $2n$ by using `eig`. In this particular case, it is known that $Q(a) < 0$ for $a = 2$. Therefore, the symmetric LEP (4.7) is a generalized symmetric definite eigenproblem.

	solver	$n = 200$	$n = 400$	$n = 600$	$n = 800$
Trimmed LEP (4.4)	`eig`	0.0156	0.1406	1.0625	3.5469
QEP (4.5)	`polyeig`	0.7500	6.8594	24.5313	84.9063
Full sym-LEP (4.7)	`eig`	0.0781	0.5938	2.0781	5.4219
$A-\lambda B$	`eig`	0.0156	0.1406	1.0469	3.5313

It is clear that the trimmed LEP (4.4) is the most efficient linearization scheme to solve the REP (4.1). Although one can efficiently exploit the positive definiteness in the generalized symmetric definite eigenproblem (4.7) it is still slower than the trimmed LEP (4.4) due to the fact that the size of (4.7) is $2n$. By the table, we also see that the brute-force approach to convert the REP (4.1) into the QEP (4.5) and then solve it via a companion form linearization is most expensive.

Note that the matrix pair $(\mathcal{A},\mathcal{B})$ in the trimmed linearization (4.4) is symmetric tridiagonal and positive definite, the same properties as the matrices A and B. Therefore, from the last row of the previous table we see that the CPU elapsed time of solving the REP via LEP (4.4) and the one of solving the eigenvalue problem of the pencil $A-\lambda B$ are essentially the same.

Example 2. This is a numerical example for the REP (4.10) discussed in section 4.3. The size of matrices A and B is $n = 36,046$. The number of nonzeros of A is $nnz = 255,088$. B has the same sparsity as A. There are nine rational terms, $k = 9$. The matrices C_i in rational terms have two dense columns and are of rank $r_i = 2$. The pole $\sigma_i = i$ for $i = 1, 2, \ldots, k$. Our aim is to compute all eigenvalues in the interval $(\alpha,\beta) = (1, 2)$, i.e., between the first and second poles.

As we discussed in section 4, the linearization of the REP (4.10) leads to the LEP $(\mathcal{A}-\lambda\mathcal{B})z = 0$, where $\mathcal{A}$ and $\mathcal{B}$ are defined as in (4.12) with the size $n + r_1d_1 + \cdots + r_kd_k = n + 2\times 9 = n + 18$.

By the expression (4.13), we can conclude that there are 8 eigenvalues in the interval. To apply the expression (4.13), we need to know the inertias of the matrices $\mathcal{A}-\tau\mathcal{B}$ for $\tau = \alpha,\beta$. These can be computed using the $\mathrm{LDL}^{\mathrm{T}}$ decomposition of the matrix $\mathcal{A}-\tau\mathcal{B}$. Since C_i are dense, the explicit computation of the $\mathrm{LDL}^{\mathrm{T}}$ decomposition of $\mathcal{A}-\tau\mathcal{B}$ is too expensive. Instead, we can first perform the following congruence transformation

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\mathcal{A}-\tau\mathcal{B}} & \\ & & -I \end{array}\right] = \mathcal{L}_1 \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A-\tau B} & C \\ & -\tau\Sigma^{-1} & I \\ \hline C^T & I & -I \end{array}\right] \mathcal{L}_1^T \tag{5.1}$$

$$= \mathcal{L}_1\mathcal{L}_2 \begin{bmatrix} A-\tau B & & \\ & -\tau\Sigma^{-1} & \\ & & F \end{bmatrix} \mathcal{L}_2^T\mathcal{L}_1^T,$$

where $F = -I - C^T(A-\tau B)^{-1}C + \Sigma/\tau$,

$$\mathcal{L}_1 = \left[\begin{array}{cc|c} I & & C \\ & I & I \\ \hline & & I \end{array}\right], \quad \mathcal{L}_2 = \left[\begin{array}{cc|c} I & & \\ & I & \\ \hline C^T(A-\tau B)^{-1} & -\tau^{-1}\Sigma & I \end{array}\right].$$

Under these congruence transformations, we see that the inertias of the matrices $\mathcal{A}-\tau\mathcal{B}$ can be computed from the the inertias of the matrices $A-\tau B$, $-\tau^{-1}\Sigma$ and F. Therefore, we only need to compute the LDL$^{\mathrm{T}}$ decomposition of the sparse symmetric matrix $A-\tau B$. In MATLAB , the LDL$^{\mathrm{T}}$ decomposition of a sparse symmetric matrix is computed by the function `ldlsparse`, which is based on [4].

Let us turn to compute the 8 eigenvalues in the interval (α, β) by applying MATLAB 's sparse eigensolver `eigs` for the LEP (4.12). The function `eigs` is based on the implicitly restarted Arnoldi (IRA) algorithm [12]. The following parameters are used for applying the function `eigs` with the shift-and-invert spectral transformation:

```
tau = 1.5;            % the shift
num = 8;              % number of wanted eigenvalues
opts.issym = true;
opts.isreal = true;
opts.disp = 1;
opts.tol = 1.e-13;    % residual bound
opts.p = 4*num;       % number of Lanczos basis
```

Furthermore, we need to provide an external linear solver for the linear system

$$\left(\begin{bmatrix} A-\tau B & \\ & -\tau\Sigma^{-1} \end{bmatrix} + \begin{bmatrix} C \\ I \end{bmatrix}[C^T\ I]\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{5.2}$$

The following procedure is such a solver to compute the solution vectors x_1 and x_2 based on the LDL$^{\mathrm{T}}$ decomposition $A-\tau B = LDL^T$ and the Sherman-Morrison-Woodbury formula[1]:

1. $C := L^{-1}C$
2. $b_1 := L^{-1}b_1$
3. $e = C^TD^{-1}b_1 - \Sigma b_2/\tau$
4. $e := (I + C^TD^{-1}C - \Sigma/\tau)^{-1}e$
5. $x_1 = L^{-T}D^{-1}(b_1 - Ce), x_2 = -\Sigma(b_2 - e)/\tau$.

The linear system (5.2) needs to be solved repeatedly for different right-hand sides. For computational efficiency, the matrix $C := L^{-1}C$ in step 1 and the matrix $(I + C^TD^{-1}C - \Sigma/\tau)^{-1}$ in step 4 are computed only once and stored before calling `eigs`.

The CPU elapsed time is displayed in the following table, where "`ldlsparse`" is the time for computing the decomposition $A-\tau B = LDL^T$. "preprocessing" is for computing the matrices $C := L^{-1}C$ and $(I + C^TD^{-1}C - \Sigma/\tau)^{-1}$, and the assemblage of the matrix $\mathcal{B} = \mathrm{diag}(B, \Sigma^{-1})$.

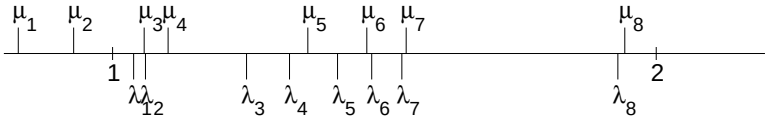
	$\mathcal{A}-\lambda\mathcal{B}$	$A-\lambda B$
`ldlsparse`	0.64	0.64
preprocessing	0.49	0
`eigs`	7.14	6.66
Total	8.27	7.30

The residuals for all 8 computed eigenpairs are less than 5.5×10^{-13}. For comparison, in the third column of the previous table, we also record the CPU time to solve only the eigenvalue problem of

[1] $(A+BB^T)^{-1} = A^{-1} - A^{-1}B(I+B^TA^{-1}B)^{-1}B^TA^{-1}$.

the linear term $A - \lambda B$ of REP (4.10) using the same parameters for calling `eigs`. As we can see, it only takes about 13.3% extra time to solve the full REP than the simple eigenvalue problem of the pencil $A - \lambda B$.

The computed 8 eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_8$ of the REP (4.10) in the interval $(1, 2)$ are plotted below, along with the computed 8 eigenvalues $\mu_1, \mu_2, \ldots, \mu_8$ of the linear pencil $A - \mu B$ closest to the shift $\tau = 1.5$.



We observe that there are only 6 eigenvalues of the pencil $A - \lambda B$ in the interval $(1, 2)$. Among them, μ_6, μ_7, μ_8 are good approximations of $\lambda_6, \lambda_7, \lambda_8$, respectively. If we treat the REP (4.10) as a general NEP and use an iterative method (such as the nonlinear Arnoldi method [22]) with the initial approximations μ_6, μ_7, μ_8, we would expect good convergence to the desired eigenvalues $\lambda_6, \lambda_7, \lambda_8$. It would be difficult to predict the convergence behavior of the iterative method with the initial approximations $\mu_1, \mu_2, \ldots, \mu_5$.

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