Bernd Hamann (2026) --- A Non-standard Representation of Chords and an Unconventional Approach Supporting the Study and Memorization of the Harmonies of the First Movement of Sonata Number 14, Opus 27, Number 2 ("Moonlight Sonata") in C#-Minor by Ludwig van Beethoven, Including Notes

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Notation: Abandoning '#': 12 Half-Tones = cCdDefFgGaAb, e.g., C#-Min = CDeFGab
Abandoning 'b': 12 Half-Tones = CdDeEFgGaAbB, e.g., F-Maj = FGAbCDE
Notes = Letters with Space in-between, e.g., C e G = C#, E, G# (in C#-Min)
Chords = Words (no Spaces between Letters), e.g., CeG = Cmin = (C#EG#)
Blue = Right Number/Index in Front of a Note = Octave on 88-key Piano the Note Is in
Red = Left → 88 Keys: 0A, 0A#, 0B, 1C, 1C#, 1D, …, 3B, 4C=midC, 4C#, …, 7A#, 7B, 8C

First Movement – Adagio Sostenuto Melody Notes in Green (Mel)

1	3G C e	G C e	G C e	G C e	
i-VII:	2CC				C#m
CeG	3G C e	G C e	G C e	G C e	
DFa	1bb				C#m7
eGb	3a C e	a C e	3a d F	a d F	
FaC	1aa		1FF		A, D (Neapolitan)
4	3G c F	G C e	G C D	F c D	
GbD	1GG		1GG		G#7, C#m, C#s2, G#7
aCe	3e G C	3G C e	G C e	Mel→ 3GG C e G	
bDF	2CGC			Motif GGG	C#m
	3GG D F	G D F	G D F	GG D F G	
	2cGc				G#7
7	3GG C e	G C e	aa C F	a C F	
	2CC		1FF		C#m, F#m
	3GG b e	G b e	aF b D	3a4b 3b D	
	1bb * 8: 3a4b Dissonant *		1bb		E, B7

	3Ge b e	G b e	G b e	G b e
	2ee * 9: E as Gbe, Gbe, Gbe, Gbe; 46: E as Gbe, beG, beG, beG *			E
10	3g b e	g b e	g b e	gg b e g
	2ee * Bars 8-12: E → B7 → E → Em → G7 → C *		Motif ggg	Em
	3gg b f	g b f	g b f	gg b f g
	2dd			G7
	3gg c e	g b e	g C e	3FF C e
	2cc	1bb	1AA	C, Em, C#fd7, F#7
13	3FF 3b d	3F b d	3gg 3b 4C	3ee 3b 4C
	1bb		2e	2g Bm, C#hd7
	3FF 3b d	3F b d	3FF A 4C	3F A 4C
	2F		1FF	Bm, F#
	3b d F	b d F	b D F	3bb 4D F
	1bb * 16, 18: 3b4A Dissonant *			Motif bbb Bm, B
16	3bb 4e g	3b e g	b e g	3b4A e g
	1bb	2ee	2gg	2ee Em, Em&A
	3bb 4D F	3b D F	b D F	3bb D F
	1bb			B
18=16	3bb 4e g	3b e g	b e g	3b4A e g
	1bb	2ee	2gg	2ee Em, Em&A
19	3bb 4D F	3b D F	3bb d f	3b d f
	1bb		1GG	B, Bfd7

Bars 20-22: Downward Melody/Motif 4b 4a 4g 4F 4C

20 3bbC G 3b C G 3aa C F 3a C F

1ff * Via C#7 to F#m * 1FF C#7, F#m

3gg b d 3g b d 3FF a D 3F a D

1bb * F#fd7 Emphasizing F#m * 2cc G, F#fd7

22 3CC F a 3C F a 3CC F G 3CC f G

2C 2C F#m, F#s2, C#

3F a C 3a C F 4C F a Mel→ 4CC F a C

1FCF * Modulation C#m → F#m, GCe → CFa * Motif CCC F#m

4CC 4G b 4C G b 4C G b 4CC G b C

2fCf * 25-28: Mirror Melody/Motif (5C) 5c 5C 5D 5e 5D 5C 5c * C#7

25 4CC 4F a 4C F a 4cc F a 4CC F a

2FF * Four Triplets Ending F a * 2DD 2CC F#m, F#fd7, F#m

4DD 4F G 4D F G 4D F G 4DD F G

2cGc * Four Triplets Ending F G / Bar 6 Chords * 2cGc G#7

4ee 4G C 4e G C 4DD F a 4CC e A

2CGC * F#m → G#7 → C#m * 1FF (F##=g) 1gg C#m, D#dim, C#fd7

Note. All diminished three-note chords are "sub-chords" of the three possible fully diminished 7th chords, i.e., the chords (C Eb Gb Bbb) = (C Eb Gb A), (C# E G Bb), (D F Ab Cb) = (D F Ab B). Using indices for the notes (C=0, C#=1, …, B=11), index tuples are (CEbGbA) = (0 3 6 9), (C#EGBb) = (1 4 7 10) and (DFAbB) = (2 5 8 11). Important is the fact that all these 7th chords consist of stacked minor thirds. Diminished three-note chords have associated half-note index sets that are index sub-sets of the 7th chord index sets {0, 3, 6, 9}, {1, 4, 7, 10} and {2, 5, 8, 11]. Thus, the three-

note chord index sets are {0, 3, 6}, {0, 3, 9}, {0,6,9}, {3,6,9}; {1,4,7}, {1,4,10}, {1,7,10}, {4,7,10}; and {2,5,8}, {2,5,11}, {2,8,11}, {5,8,11}. Further, each of these three-note chords can be played in six (3!=6) positions. One must memorize these diminished three-note chord sequences (where B# = c):

28 5c 4c 4D 4G 4c 4D 4a 4c 4D 4F 4c 4D 8x cD

GG = 1GG * 28-41: No Recognizable Melody/Motif * G#, Adim, G#7

Ped. ----- 4c 4D 3G 4c 4D 3a 4c 4D 3F 4c 4D

Pnt. 1GG * 29: 1st Triplet Note 1 Oct. Lower than in 28 * G#, Adim, G#7

(28-40) 3e 4e 4G 5C 4e 4G 5e 4e 4G 5C 4e 4G 8x eG

1GG C#m

31 ----- 3e 3G 4C 3e 3G 4e 3e 3G 4C 3e 3G

1GG * 31: 1st Triplet Note 1 Octave Lower than in 30 * C#m

Note. Three-note diminished chords are sub-chords of the fully diminished (FD) 7th chords FD1, FD2, or FD3, i.e., the 7th chords

FD1 = (C[B#] Eb Gb Bbb[A]) = (C[B#] D# F# Bbb[A]),

FD2 = (C# E G Bb) = (Db E G A#), or

FD3 = (D F Ab Cb[B]) = (D F G# Cb[B]).

The sequences of three-note diminished chords that include Cdim, D#dim, F#dim and Adim (bars 32, 35, 36, 37) use sub-chords of FD1. The sequence of three-note diminished chords that includes C#dim, Edim, and Gdim (bar 34) uses sub-sets of FD2. This fact makes is easier to memorize the following sequences:

Bar 32: FD1 Progression from 3D to 5c – 2-up-1-down

Bar 33: C#m Triad Notes from 3e to 5C – 2-up-1-down (to 4G)

Bar 34: FD2 Progression from 4C to 5A – 2-up-1-down

Bar 35: FD1 Progression from 4F to 6D – 2-up-1-down

Bar 36: FD1 Progression from 6c to 4D – 2-down-1-up

Bar 37: FD1 Progression from 4F to 3D – 2-down-1-up (to 3F)

NO 3D 3a 3F 4c 3a 4D 4c 4F 4D 4a 4F 5c

ME- 1GG * 32: Dimin. Chord Progress. * D#dim, Adim, Cdim, F#dim - FD1 UP

LO- 3e 4C 3G 4e 4C 4G 4e 5C 4G 5e 5C 4G

DY 1GG * 33: Fourth Triplet downward e C G * C#m

34 4C 4g 4e 4A 4g 5C 4A 5e 5C 5g 5e 5A

IN 1GG * 34-37: Dim. Chord Progr. * C#dim, Gdim, A#dim, Edim - FD2 UP

32-42 4F 5c 4a 5D 5c 5F 5D 5a 5F 6c 5a 6D

1GG F#dim, Cdim, D#dim, Adim - FD1 UP

6c 5F 5a 5D 5F 5c 5D 4a 5c 4F 4a 4D

1GG * 35-37: From 6D to 3D * F#dim, Cdim, Adim, D#dim - FD1 DOWN

37 4F 4c 4D 3a 4c 3F 3a 3D 3F 3C 3F 3a

1GG * Dim. Chord Prog. to F#m * Cdim, F#dim, D#dim, F#m - FD1 DOWN

3c 3F 3G 3a 3G 3F 3D 3F 3a 3C 3F 3a

1GG G#7, D→F→a, D#dim, F#m

3c 3F 3G 3a 3G 3F 3d 3F 3a 3C 3F 3a

1GG G#7, D→F→a, D (Neapolitan), F#m

40 3c 3F 3G 3a 3G 3F 3C 3e 4C 3C 3e 3C

1GG 1aa G#7, D→F→a, A, A

41 3D 3a 4C 3D 3a 4C 3D 3G 4c 3D 3F 4c

1FF 1GG D#hd7, D#hd7, G#, G#7

Bar 41 Ending with G#7 for Modulating to C#m in Following Bar 42

Note. The notation "hd7" or "hdim7" refers to half-diminished 7th chords, and the notation "fd7" or "fdim7" refers to fully diminished 7th chords.

Bars 42-45 Repeating Bars 5-8 (Reprise) → Original Melody in Bar 42

42 3e 3G 4C 3G C e G C e Mel→ 3GG C e G

2CGC Motif GGG C#m

43 3GG D F G D F G D F GG D F G

2cGc G#7

3GG C e G C e aa C F a C F

2CC 1FF C#m, F#m

3GG b e G b e aF b D 3a4b 3b D

1bb * 45: 3a4b Dissonant * 1bb E, B7

46 3Ge b e 3b e G b e G 3bb e G b

2ee * 45-48: E → B7 → E → B7 → E (see 8-12) * Motif bbb E

47 3bb F a 3b F a b F a 3bb F a b

2DD * 48-51: Loop Melody/Motif (4b) 5c 5C 5D 5e 5d 5c (5C) * B7

48 3bb e G 3b e G 4cc F G 4CC e G

2ee * 46-48: Bass 3e 3D 3e 3D * 2DD E, G#7, C#m

49 4DD F G 4D F G 4ee G C 4e G C

2cGc * G#7 → C#m (in 26, 27) * 2CGC G#7, C#m

4dd F a 4d F a 4cc F G 4c F G

1FF 1GG D (Neapolitan), G#7

Bar 51 = Bar 15 Raised by Two Half-tones: Bm → C#m and B → C#

51 4CC e G 4C e G 4C f G 4CC f G {C}

2CC * 51-54: {C}, {c} Implied-not-played * Motif CCC C#m, C#

16-19: Em → Em&A → B; 52-55: F#m&D → F#m → F#m&C → C#

Bass Chords in 16-19: bb-ee-gg-ee-bb; in 52-55: CC-FF-aa-FF-CC

Dissonant Chords 4Cd, 4Cc (52, 54) and 3FG (59)

52	4Cd F a	4C F a	4C F a	4Cc F a {c}
C =	2CC	2FF	2aa	2FF F#m&D, F#m, F#m&C
Ped.	4CC f G	4C f G	4C f G	4CC f G {C}
Pnt.	2CC * 52, 54: 4Cd, 4Cc Dissonant *			C#
(52-55)	4Cd F a	4C F a	4C F a	4Cc F a {c}
54=52	2CC	2FF	2aa	2FF F#m&D, F#m, F#m&C

Bars 55-60: Downward Melody/Motif 5C 4b 4a 4G 4F (4G 4a 4G) 4C

55	4CC f G	4C f G	4CC F a	4C F a
	2CC		2FF	C#, F#m
	3bb F a	3b F a	3b F a	3bb e G
	2DD			2ee B7, E
57	3aa e G	3aa D F	3GG D F	3GG C e
	2CC	2DD	2cc	2CC
	* Bar 57: Loop Bass 3C 3D 3c 3C *			Amaj7, D#dim, G#7, C#m
58	3FF C D	3F C D	3GG C D	3aa C D
	1aa * Four Triplets Ending C D *		1GG	1FF D#hd7, C#s2

Bars 59-67: 1st Triplet Two-note Chords in Left Hand: GG, CG, and cG

	3GG C e	3G C e	3FG c D	3F c D
	1GG * 59: 3FG Dissonant *		1GG	C#m, G#7, Cdim

60 3eC 3G C 3G C e 3G C e 3G C e

2CG * Melody/Motif now in Left Hand * Mel → 2G 2G

Motif GGG C#m

61 3G D F 3G D F 3G D F 3G D F

2cG 2G 2G G#7

62 3G 4e 4C 4G 4e 5C 4G 5e 5C 5G 5e 5C

2CG * Bars 62, 64 and 66: C#m Triad Notes * 2G 2G C#m

63 5c 5D 4a 5c 4F 4a 4D 4F 3a 4c 3a 3G 3F

1GG * 63: Dim. Chord Progr.: 5c 5D 4a ... 4F 3a * 2G 2G

G, G = "Pedal Pnt." Cdim/G#, F#dim, D#dim, Cdim/G# - FD1 DOWN

64=62 3eC 4e 4C 4G 4e 5C 4G 5e 5C 5G 5e 5C

2CG * 62: G=1st Note, 64: eC = 1st [Chord] Notes * 2G 2G C#m

65=63 5c 5D 4a 5c 4F 4a 4D 4F 3a 4c 3a 3G 3F

1GG 2G 2G

G, G = "Pedal Pnt." Cdim/G#, F#dim, D#dim, Cdim/G# - FD1 DOWN

66 3eC 3G 4C 4e 4C 3G [3C] 3e 3G 4C 3G 3e

2CG * 2nd Invers.: GCeCG * 3C * 1st Invers.: eGCGe * C#m

67 [2CG]3C 3e 3G 3e 3C [2G 3C 2G 2e 2G 2e]

2CG * Root Pos.: CeGeC * 2G 3C 2G 2e 2G 2e C#m

68 [2C] ---------- 3eGC ----------

2C * 67,68 Bass: GCG, eGe, C * 2CGC ---------- C#m

69 3eGC ||

2CGC C#m ||

Note. [...] = Playing-Notes-

-by-Right-Hand Alternative

Triads, Seventh Chords, Other Chords (Root Position, Inversions)

--- Using Chord Positions Having F#, G, G# or A as Highest Notes

1	C#m	C# E G#		
2	C#m7	B C# E G#		
3	A	C# E A	D Neapolitan	D F# AD not in C#m
4	G#7	B#(C) D# F# G#	C#m	C# E G#
	C#s2	C# D# G#	G#7	B#(C) D# F# G#
5	C#m	C# E G#		
6	G#7	B#(C) D# F# G#		
7	C#m	C# E G#	F#m	A C# F#
8	E	B E G#	B7	A B D# F#
9	E	B E G#		
10	Em	B E G		
11	G7	B D F G G not in C#m		
12	C	C E G C in harm. C#m	Em	B E G
	C#fd7	A# C# E G	F#7	A# C# E F#
13	Bm	B D F#	C#hd7	B C# E G
14	Bm	B D F#	F#	A# C# F#
15	Bm	B D F#	B	B D# F#
16	Em	B E G	Em & A#	A# B E G
17	B	B D# F#		
18	Em	B E G	Em & A#	A# B E G
19	B	B D# F#	Bfd7	B D F G#
20	C#7	B C# E#(F) G#	F#m	A C# F#
21	G	B D G G not in C#m	F#fd7	A C D# F#
22	F#m	A C# F#	F#s2	C# F# G#
	C#	C# E#(F) G#		

23	F#m	A C# F#		
24	C#7	B C# E#(F) G#		
25	F#m	A C# F#	F#fd7	A C D# F#
	F#m	A C# F#		
26	G#7	G# B#(C) D# F#		
27	C#m	C# E G#	D#dim	A D# F#
	C#fd7	A# C# E G		
28	G#	B#(C) D# G#	Adim	C D#(Eb) A
	G#7	C D# F# G#		
29	G#	B#(C) D# G#	Adim	C D#(Eb) A
	G#7	C D# F# G#		
30	C#m	C# E G#		
31	C#m	C# E G#		
32	D#dim	D# F#(Gb) A	Adim	C D#(Eb) A
	Cdim	C D#(Eb) F#(Gb)	F#dim	A C F#
33	C#m	C# E G#		
34	C#dim	C# E G	Gdim	A#(Bb) C#(Db) G
	A# dim	C# E A#	Edim	E G A#(Bb)
35	F#dim	A C F#	Cdim	C D#(Eb) F#(Gb)
	D#dim	D# F# A	Adim	C D#(Eb) A
36	F#dim	A C F#	Cdim	C D#(Eb) F#(Gb)
	Adim	C D#(Eb) A	D#dim	D# F# A
37	Cdim	C D#(Eb) F#(Gb)	F#dim	A C F#
	D#dim	D# F# A	F#m	A C# F#
38	G#7	B#(C) D# F# G#	A → G# → F#	
	D#dim	D# F# A	F#m	A C# F#

39	G#7	B#(C) D# F# G#	A → G# → F#	
	D Neapolitan	D F# A D not in C#m	F#m	A C# F#
40	G#7	B#(C) D# F# G#	A → G# → F#	
	A	C# E A	A	C# E A
41	D#hd7	C# D# F# A	D#hd7	C# D# F# A
	G#	B#(C) D# G#	G#7	B#(C) D# F# G#
42	C#m	C# E G#	C#m	C# E G#
	C#m	C# E G#	C#m	C# E G#
43	G#7	B#(C) D# F# G#	G#7	B#(C) D# F# G#
	G#7	B#(C) D# F# G#	G#7	B#(C) D# F# G#
44	C#m	C# E G#	C#m	C# E G#
	F#m	A C# F#	F#m	A C# F#
45	E	B E G#	E	B E G#
	B7	A B D# F#	B7	A B D# F#
46	E	B E G#	E	B E G#
	E	B E G#	E	B E G#
47	B7	A B D# F#	B7	A B D# F#
	B7	A B D# F#	B7	A B D# F#
48	E	B E G#	E	B E G#
	G#7	B#(C) D# F# G#	C#m	C# E G#
49	G#7	B#(C) D# F# G#	G#7	B#(C) D# F# G#
	C#m	C# E G#	C#m	C# E G#
50	D Neapolitan	D F# A D not in C#m	D Neapolitan	D F# AD not in C#m
	G#7	B#(C) D# F# G#	G#7	B#(C) D# F# G#
51	C#m	C# E G#	C#m	C# E G#
	C#	C# E#(F) G#	C#	C# E#(F) G#

52	F#m & D	A C# D F#	F#m	A C# F#
	F#m	A C# F#	F#m & C	A C C# F#
53	C#	C# E#(F) G#	C#	C# E#(F) G#
	C#	C# E#(F) G#	C#	C# E#(F) G#
54	F#m & D	A C# D F#	F#m	A C# F#
	F#m	A C# F#	F#m & C	A C C# F#
55	C#	C# E#(F) G#	C#	C# E#(F) G#
	F#m	A C# F#	F#m	A C# F#
56	B7	A B D# F#	B7	A B D# F#
	B7	A B D# F#	E	B E G#
57	Amaj7	A C# E G#	D#dim	D# F# A
	G#7	B#(C) D# F# G#	C#m	C# E G#
58	D#hd7	C# D# F# A	D#hd7	C# D# F# A
	C#s2	C# D# G#	D#hd7	C# D# F# A
59	C#m	C# E G#	C#m	C# E G#
	G#7	B#(C) D# F# G#	Cdim	C D#(Eb) F#(Gb)
60	C#m	C# E G#	C#m	C# E G#
	C#m	C# E G#	C#m	C# E G#
61	G#7	B#(C) D# F# G#	G#7	B#(C) D# F# G#
	G#7	B#(C) D# F# G#	G#7	B#(C) D# F# G#
62	C#m	C# E G#	C#m	C# E G#
	C#m	C# E G#	C#m	C# E G#
63	Cdim/G#	C D#(Eb) F#(Gb) / G#	F#dim	A C F#
	D#dim	A D# F#	Cdim/G#	C D#(Eb) F#(Gb) /G#
64	C#m	C# E G#	C#m	C# E G#
	C#m	C# E G#	C#m	C# E G#

65	Cdim/G#	C D#(Eb) F#(Gb) / G#	F#dim	A C F#
	D#dim	A D# F#	Cdim/G#	C D#(Eb) F#(Gb) /G#
66	C#m	C# E G#	C#m	C# E G#
	C#m	C# E G#	C#m	C# E G#
67	C#m	C# E G#	C#m	C# E G#
	C#m	C# E G#	C#m	C# E G#
68	C#m	C# E G#	C#m	C# E G#
69	C#m	C# E G#	Piano Octave: _^_^_ _ ^_^_^_ _ →	

Chord (2C#, 2G#, 3C#, 3E, 3G#, 4C#): _2C#_^_ _ ^_2G#_^_ _3C#_^3E_^_3G#_^_ _4C#_^_ _^_^_^_ _

The triplet chord sequence can be visualized by indicating the notes played with the left and right hands within only one reference octave.

Triplet Chord Visualization in One Octave C(0), C#/Db(1), ..., B(11):

```
      1       3           6       8       10
  0       2       4   5       7       9       11
    C#/Db   D#/Eb       F#/Gb   G#/Ab   A#/Bb
    |   |   |   |       |   |   |   |   |   |
|   |   |   |   |   |   |   |   |   |   |   |   |
  C       D       E   F       G       A       B
```

Visualization of four Notes of Triplet Chord C#m7 = (C#,E,G#,B):

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      *               *

              *               *
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Left (L) & Right (R) Hands Playing Black (BL) & White (WH) Keys of C#m7:

```
      R               R               BL
      L       R               R       WH
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	L R			R				R			R
C	C#/Db	D	D#/Eb	E	F	F#/Gb	G	G#/Ab	A	A#/Bb	B

C#-Minor Scale-inherent Triads Using Roman Numeral Indexing:

i	= C#m = C#EG#	iid	= D#dim = D#F#A	III	= E = EG#B
iv	= F#m = F#AC#	v	= G#m = G#BD#		
VI	= A = AC#E	VII	= B = BD#F#		

Circle-of-Fifths Neighborhood of Keys with Three-Note Chords

Sevenths	C7	G7	D7	A7	E7	B7
Triads	CEG	GBD	DF#A	AC#E	EG#B	BD#F#
Major	C	G	D	A	E	B
Sevenths	C7	G7	D7	A7	E7	B7
Diminished		F#AC			D#F#A	
Minor	Am	Em	Bm	F#m	C#m	G#m
Triads	ACE	EGB	BDF#	F#AC#	C#EG#	G#BD#
Sevenths	Am7	Em7	Bm7	F#m7	C#m7	G#m7

All Circle-of-Fifths Neighborhoods Used for Progressions

(C = Cmaj, a = Amin, b" = Bdim, F+ = F#. B- = Bb etc.)

G-	D-	A-	E-	B-	F	C	G	D	A	E	B	F+
G-B-D-F					FACE	CEGB	GBDF+					F+A+C+E+
f"	c"	g"	d"	a"	e"	b"	f+"	c+"	g+"	d+"	a+"	e+"
fa-c-e-					egb-d	bdfa	f+ace					e+g+bd+
e-	b-	f	c	g	d	a	e	b	f+	c+	g+	d+
e-g-b-d-					dfac	aceg	egbd					d+f+a+c+

One can "smoothly move" left to right, right to left, up or down.

Relevant Scales of Relative Major and (Harmonic) Minor Keys:

Minor	i	iidim	III	iv	v	VI	VII (viidim)		
Major			I	ii	iii	IV	V	vi	viidim
F#m, A:	F#	G#	A	B	C#	D	E (E#=F)	F#	G#
C#m, E:	C#	D#	E	F#	G#	A	B (B#=C)	C#	D#
G#m, B:	G#	A#	B	C#	D#	E	F# (F##=G)	G#	A#

Chords can be defined by selection of indices/notes from an index/note set { {0}, {2,3,4,5}, {6,7,8}, {9,10,11}, {13}, {17}, {21} }. Specifically, if one chooses 0 = C, i.e., using C as root of chords, the set is { {C}, {Ebb,Eb,E,E#}, {Gb,G,G#}, {Bbb,Bb,B}, {D},{F},{A} }. By selecting either zero or one note from each note subset, many possible chords with root C --- and their inversions --- result:

{}

C

C Ebb

C Eb

...

C A

C Ebb Gb

C Ebb G

...

C E# G

...

C F A

C Ebb Gb Bbb

C Ebb Gb Bb

C Ebb Gb B

...

C E G B

...

C Ebb Gb Bbb D

...

C E# G# B D F A

One can define a set of index sets more compactly as

$\{ S_0, S_1, S_2, S_3, S_4, S_5, S_6 \} = \{ S0, S25, S68, S911, S14, S17, S21 \}$

$= \{ \{0\}, \{2,3,4,5\}, \{6,7,8\}, \{9,10,11\}, \{14\}, \{17\}, \{21\} \}$

$= \{ \{R\}, \{R+2,R+3,R+4,R+5\}, \{R+6,R+7,R+8\},$
$\{R+9,R+10,R+11\}, \{R+14\}, \{R+17\}, \{R+21\} \}.$

Using this set definition, given a specific note (or its index, an integer value between 0 and 11), all possible chords built on it are obtaoined by selecting a note (or not) from the set sequence S0 → S25 → S68 → S911 → S14 → S17 → S21.

Abstractly, one can represent the construction of a chord as a "combination" or "concatenation" of up to seven notes, where each note N is selected from one of the seven note subsets, i.e.,

$$\begin{matrix} & 2 & & & & & \\ & 3 & & 6 & & 9 & \\ 0 & + 4 & + 7 & & + 10 & + 14 + 17 + 21 & = (N1,N2,N3,N4,N5,N6,N7). \\ & 5 & & 8 & & 11 & \end{matrix}$$

Based on this general index-/step-based notation, every note of the chromatic scale of 12 half-tones can serve as root (index) r. Considering the C-major scale C-D-E-F-G-A-B, with associated indices/steps 0-2-4-5-7-9-11, each of these seven notes can be the root (index) r, defining the possible chords (and inversions) with root r. One can use the resulting 1-, 2-, ..., 7-note chords (and inversions) to determine a multitude of possible smooth (or non-smooth) progressions, with chords moving with minimal, small, large of maximal note changes from chord to chord. Beethoven employs such smooth progressions, involving nearly all possible major, minor, diminished, suspended or augmented triads and 7^{th} chords, using C#-minor scale notes as chord roots.

TRIADS OF REL. MIN & MAJ				→	SEVENTH CHORDS		STACKS
C#min Scale		Emaj Scale			Concatenation		m, M 3rds
i =	C#EG#	I =	EG#B	→	i + I =	C#EG#B	mMm
iid =	D#F#A	ii =	F#AC#	→	iid + ii =	D#F#AC#	mmM
III =	EG#B	iii =	G#BD#	→	III + iii =	EG#BD#	MmM
iv =	F#AC#	IV =	AC#E	→	iv + IV =	F#AC#E	mMm
v =	G#BD#	V =	BD#F#	→	v + V =	G#BD#F#	mMm
VI =	AC#E	vi =	C#EG#	→	VI + vi =	AC#EG#	MmM
VII =	BD#F#	viid=	D#F#A	→	VII + viid=	BD#F#A	Mmm

The concatenation of the scale-inherent triads of the C#min and Emaj scales yield seven seventh chords consisting of stacked thirds, which are minor (m) or major (M) thirds. The types of stacks of three thirds defining the seven seventh chords are: mMm (3x), mmM (1x), MmM (2x), and Mmm (1x).

Sixth chords. The following augmented sixth chords are also relevant:

- **Italian Sixth: It contains the lowered 6th, the tonic, and the raised 4th, e.g., (A♭ C F♯) in Cmaj.**
- **French Sixth: It adds the 2nd scale degree to the Italian, e.g., (A♭ C D F♯) in Cmaj.**
- **German Sixth: It adds the 3rd scale degree to the Italian, e.g., (A♭ C E♭ F♯) in Cmaj.**

Chords and Their Inversions for the C#m Scale C# D# E F# G# A B and the Relative E Scale E F# G# A B C# D#

			I, vi						iid, viid			
			C#	E	G#	“B”			D#	F#	A	“C#”
		“B”	C#	E	G#			“C#”	D#	F#	A	
	G#	“B”	C#	E			A	“C#”	D#	F#		
E	G#	“B”	C#			F#	A	“C#”	D#			

			III, I						iv, ii			
			E	G#	B	“D”			F#	A	C#	“E”
		“D”	E	G#	B			“E”	F#	A	C#	
	B	“D”	E	G#			C#	“E”	F#	A		
G#	B	“D”	E			A	C#	“E”	F#			

			V, iii						VI, IV			
			G#	B	D#	“F#”			A	C#	E	“G”
		“F#”	G#	B	D#			“G”	A	C#	E	
	D#	“F#”	G#	B			E	“#”	A	C#		
B	D#	“F#”	G#			C#	E	“#”	A			

			VII, V			
			B	D#	F#	“A”
		“A”	B	D#	F#	
	F#	“A”	B	D#		
D#	F#	“A”	B			

<u>Notation.</u> **“N” represents the notes N#, N and Nb. Thus, “B” = B#, B or Bb; “C#” = C##, C# or C; “D” = D#, D or Db; “E” = E#, E or Eb; “F#” = F##, F# or F; “G” = G#, G or Gb; and “A” = A#, A or Ab. These three notes define the fourth note of the various seventh chords.**

"Embedding" a Melody Note, e.g., Note C, in Three-note Chords, in Major, Minor, Suspended, Diminished and Augmented Chords

Chord	Notes of Three-note Chord					Three Chord Positions
C aug			C	E	Ab	C E Ab, E Ab C, Ab C E
Ab aug		Ab	C	E		Ab C E, C E Ab, E Ab C
E aug	E	Ab	C			E Ab C, Ab C E, C E Ab
C sus4			C	F	G	C F G, F G C, G C F
G sus4		G	C	D		G C D, C D G, D G C
F sus4	F	Bb	C			F Bb C, Bb C F, C F Bb
C maj			C	E	G	C E G, E G C, G C E
Ab maj		Ab	C	Eb		Ab C Eb, C Eb Ab, EbAbC
F maj	F	A	C			F A C, A C F, C F A
C min			C	Eb	G	C Eb G, Eb G C, G C Eb
A min		A	C	E		A C E, C E A, E A C
F min	F	Ab	C			F Ab C, Ab C F, C F Ab
C sus2			C	D	G	C D G, D G C, G C D
Bb sus2		Bb	C	F		Bb C F, C F Bb, F Bb C
F sus2	F	G	C			F G C, G C F, C F G
C dim			C	Eb	Gb	C Eb Gb, EbGbC, GbCEb
A dim		A	C	Eb		A C Eb, C Eb A, Eb A C
Gb dim	Gb	A	C			Gb A C, A C Gb, C Gb A

Note. Analogously, one can "embed" a melody note in 7th chords.

"Embedding" a Melody Note, e.g., Note C, in Four-note Chords,
in CM7 = Cmaj7, C7, Cm7, Chd7 = Chalf-dim7, Cfd7 = Cfull-dim7

Chord	Notes of Four-note Chord					Three Chord Inversions
CM7			C	E	G	B	EGBC, GBCE, BCEG
AbM7		Ab	C	Eb	G		CEbGAb, EbGAbC, GAbCEb
FM7	F	A	C	E			ACEF, CEFA, EFAC
DbM7 Db	F	Ab	C				FAbCDb, AbCDbF, CDbFAb
C7			C	E	G	Bb	EGBbC, GBbCE, BbCEG
Ab7		Ab	C	Eb	Gb		CEbGbAb, EbGbAbC, GbAbCEb
F7	F	A	C	Eb			ACEbF, CEbFA, EbFAC
D7 D	F#	A	C				F#ACD, ACDF#, CDF#A
Cm7			C	Eb	G	Bb	EbGBbC, GBbCEb, BbCEbG
Am7		A	C	E	G		CEGA, EGAC, GACE
Fm7	F	Ab	C	Eb			AbCEbF, CEbFAb, EbFAbC
Dm7 D	F	A	C				FACD, ACDF, CDFA
Chd7			C	Eb	Gb	Bb	EbGbBbC, GbBbCEb, BbCEbGb
Ahd7		A	C	Eb	G		CEbGA, EbGAC, GACEb
F#hd7	F#	A	C	E			ACEF#, CEF#A, EF#AC
Dhd7 D	F	Ab	C				FAbCD, AbCDF, CDFAb
Cfd7			C	Eb	Gb	Bbb	EbGbAC, GbACEb, ACEbGb
Afd7		A	C	Eb	Gb		CEbGbA, EbGbAC, GbACEb
F#fd7	F#	A	C	D#			ACD#F#, CD#F#A, D#F#AC
D#fd7 D#	F#	A	C				F#ACD#, ACD#F#, CD#F#A

Resolving Tension Chords --- Tension chords can often be resolved in several ways. For example, tension chords on the root note C can be resolved by the chords listed in this table:

Tension Chords		Resolution Chords (Triads)	
Csus2	C D G	I, i	C E G, C Eb G
Csus4	C F G	I, i	C E G, C Eb G
Cdim	C Eb Gb	I, iii, V	C E G, B E G, B D G
Caug	C E G#	I, V, vi	C E G, B D G, C E A
Cmaj7	C E G B	I	E G C
C7	C E G Bb	i, IV	C Eb G, C F A
Cmmaj7	C Eb G B	V	D G B
Cm7	C Eb G Bb	IV, V	C F A, D G B
Chdim7	C Eb Gb Bb	i, ii, iii, IV	C Eb G, D F A, E G B, C F A
Cfdim7	C Eb Gb Bbb(A)	I, i, IV, V	C E G, C Eb G, C F A, B D G

Other resolutions --- considering also resolution chords in other keys --- are possible. Further, to "complete" this list, one must keep in mind the possible two inversions of 3-note chords and the three inversions of 4-note chords.

If one wanted to devise a sequence of chords using only the seven notes at the C-major scale, including both resolved and rension chords oin the sequence, one could base such a sequence on the tension chords Csus2, Csus4 and Cmaj7 and the resolution chords I = Cmaj, IV = Fmaj, V = Gmaj, ii = Dmin, iii = Emin, vi = Amin --- and viidim = Bdim. Using just these chords makes it possible to construct many interesting and "smooth" chord progressions.

"Smooth" Scale C#-D#-E-F#-G#-A-B with 7th Chords and Triads:

C#m7	BEG#C#	C#m	EG#C#	A	EAC#	F#m	F#AC#
D#hd7	C#F#AD#	D#d	F#AD#	B	F#BD#	G#m	G#BD#
Emaj7	D#G#BE	E	G#BE	C#m	G#C#E	A	AC#E
F#m7	EAC#F#	F#m	AC#F#	D#d	AD#F#	B	BD#F#
G#m7	F#BD#G#	G#m	BD#G#	E	BEG#	C#m	C#EG#
Amaj7	G#C#EA	A	C#EA	F#m	C#F#A	D#d	D#F#A
B7	AD#F#B	B	D#F#B	G#m	D#G#B	E	EG#B

A mathematical-computational method for determining smooth chord progressions. One can use a "chord graph" to represent all chords and their relationships, i.e., how chords relate to each other as far as tension-building and tension-resolutions are concerned. One can use half-tone indices to represent the eleven half-tones, with the index 0 being relative to a specific key's root note, e.g., Cmaj = C0,4,7 represents the chord (C E G), with C being the root (index 0); Amin = A0,3,7 represents the chord (A C E), with A being the root; and C7 = C0,4,7,10 represents the chord (C E G Bb), with C being the root. These chords (chord representations) become the nodes in a chord graph, and directed edges between nodes indicate whether moving from one chord to another is tension-building or tension-resolution. Such a graph can capture all possible progressions.

The generation of a "smooth progression from a source chord (or key) to a target chord" can mathematically also be viewed as a combinatorial optimization problem. Every possible chord can be encoded by a 12-bit binary word b0b1b2…b11, where a bit is 0

when a note (half-tone) in not played and 1 when it is played. For example, the word 100010010000 represents the chord Cmaj = CEG. In other words, a 12-bit sequence defines which note of the 12-note sequence C-C#-D-D#-...-A#-B is played. A 12-bit word can encode 2^12 = 4096 chords.

When one wants to generate an "optimally smooth progression" from a source chord, encoded by the word s0s1s2...s11, to a target chord, encoded by the word t0t1t2...t11, the goal is to determine a sequence (progression) of chords "between source and target chords" that minimizes the sum of all chord-pair differences/distances of the chords C_1, C_2, C_3, ..., C_n, with C_1 being the source chord and C_n being the target chord. Each chord pair C_i, C_i+1 in this ordered set of chords has a chord distance/difference value Di = distance_(C_i, C_i+1), i=1,...,n-1.

The definition of a "proper chord-to-chord distance/difference" measure is crucially important. For example, one could define the overall sum by counting the number of bit disagreements between chord-pair encodings to compute the value of the sum. For example, Cmaj = 100010010000 and Cmin = 100100010000 have distance 2 since their bits in positions b3 and b4 disagree.

An optimal progression from a source to a target chord would thus require one to solve a combinatorial optimization problem: One must search for a chord progression that leads to a value of the sum of chord-pair difference/distances that is minimal.

Since each chord is encoded by a 12-bit binary number, one can Interpret each chord as a corner of a hyper-cube in a space of dimension 4096. A chord progression from a source to a target chord traverses corners of this hyper-cube, and the objective is the determination of a "polyline" in 4096-dimensional space that moves from corner to corner of the hyper-cube – "smoothly." By using only edges of the hyper-cube for a single progression step, to move from C_i to C_i+1, one uses chord pairs that differ only by one note (in one position of their associated binary codes).

Note. **A circular, cyclic visualization of the half-tone notes is helpful to "see" all possible chord positions. A "flattened circle-of-12" can be used as a representation of the 12 half-tones (interpreted in clockwise and counterclockwise orientations on the "flattened circle-of-12").**

Counterclockwise ← → Clockwise

A#/Bb B C C#/Db D

A D#/Eb

G#/Ab G F#/Gb F E

Clockwise ← → Counterclockwise

Example: Seventh Chord C#m = (C#,E,G#,B) visualized on the circle

→ All eight chord positions 1-8 represented on circle, i.e.,

1: (C#,E,G#,B), 2: (E,G#,B,C#), 3: (G#,B,C#,E), 4: (B,C#,E,G#),

5: (B,G#,E,C#), 6: (C#,B,G#,E), 7: (E,C#,B,G#), 8: (G#,E,C#,B)

___ B ___ C# ___ Clockwise and

___ ___ Counterclockwise

G# ___ ___ ___ E Positions (1-4, 5-8)

Positions 1-4 Played with Right Hand in One Octave

Positions 5-8 Played with Left and Right Hands in Two Octaves

(The four 7th-chord notes can be split left – right in other ways.)