Modeling and Visualization of Uncertainty-aware Geometry using Multi-variate Normal Distributions

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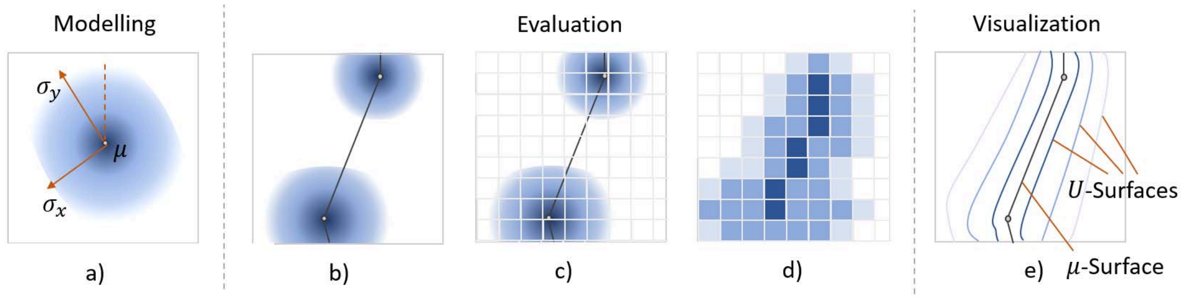

Figure 1: Modeling, evaluation and visualization of uncertain geometry. a) Modeling of uncertainty through multi-variate normal distributions. b) Uncertain geometry implied by uncertain points and lines. c) Evaluation grid with user-defined size to evaluate uncertain geometry. d) Evaluation of uncertain geometry at each grid point. e) Visualization of uncertain geometry showing the μ-surface and U-surfaces based on different iso-values.

Abstract

Many applications are dealing with geometric data that are affected by uncertainty. This uncertainty is important to analyze, visualize, and understand. We present a methodology to model uncertain geometry based on multi-variate normal distributions. In addition, we propose a visualization technique to represent a hull for uncertain geometry capturing a user-defined percentage of the underlying uncertain geometry. To show the effectiveness of our approach, we have modeled and visualized uncertain datasets from different applications.

Keywords: Modeling of Uncertainty, Uncertainty Visualization

1 Introduction

Classical geometry is concerned with objects such as points, lines and surfaces and their properties with respect to space [11]. The underlying points of a geometry are considered as an absolute groundtruth for determining the size and shape of an object. Contrary to this concept, a variety of real-world problems face the challenge that geometric descriptions cannot be evaluated exactly hence requiring us to characterize uncertainty. Examples are work pieces in mechanical engineering that contain tolerances or geometric descriptions of a patient's organs and pathologies that cannot be determined exactly as they originate from a reconstructed image [3].

The visualization of such uncertain geometry is highly desired to provide domain scientists with a better understanding of the possible shapes of a geometry resulting from the underlying uncertainty. Contrary to classic geometry, uncertain geometry cannot be visualized directly, see Section 2. The communication of the inherent uncertainty is an important step in allowing users to understand the uncertainty-affected geometry and perform decision-making based on this type of data [7]. To achieve this, a proper description of uncertainty-aware geometry is desired which can be visualized quickly and simply. Furthermore, the visualization should be consistent, which means that it should be free from geometric and topological errors. At last, the model and the resulting visualization should be general to allow for a large degree of freedom, for users to model and explore uncertainty-aware geometry.

We present a methodology to model uncertain geometry using multi-variate normal distributions instead of fixed-point descriptions. Points are expected to be located at a specific spatial position but can alter their position depending on the underlying parameters of the normal distribution. We extend the concept of uncertainty-ware points to line segments and triangles, defining an uncertain geometry. The uncertainty-aware description of a geometry can be evaluated by a grid used to visualize the uncertainty-aware geometry. We present a technique based on the isosurface concept that shows the expected geometry (μ-surface) and a covering hull, the U-surface containing all geometry fulfilling a user-defined minimal probability, see Section 3.

Therefore, this paper contributes:

- We introduce an uncertainty-aware description of geometric objects based on multi-variate normal distributions.
- We present an intuitive visualization for uncertainty-aware geometry based on extracted isosurfaces of an evaluation grid.

To show the usability of the present approach, we have applied it to datasets from fluid dynamics and medical applications, see Section 4. At last, this work is summarized and future directions are given in Section 5.

2 Related Work

This Section summarizes the state of the start in modeling and visualizing uncertain geometry. A general summary of uncertainty visualization techniques is provided in [3].

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During the last decades, several methods targeted to model uncertainty in geometry using implicit functions [15], offset surfaces [12, 13] or fuzzy sets [4]. Although these methods are able to represent a variety of geometric descriptions, the methods can lead to incorrect visualizations of geometry and/or topology. For example, they can contain self-intersections, or cannot be utilized in 3D space as they depend on concepts that cannot be generalized easily. In contrast to these methods, we present a model of uncertain geometry based on multi-variate normal distributions that can be evaluated and visualized with consistent geometry and topology.

Different methods utilize isosurfaces [2, 16, 18] that are surrounded by heatmaps indicating the probability of a surface to alter its position in space. Although this visualization can provide a good overview over the potential locations of surfaces, it can result in visual clutter. The presented method uses surrounding surfaces to indicate a hull where each possible geometry is located in.

Drapikowski [5] described a model for isosurface uncertainty chracterization in medical applications based on geometric features such as smoothness and curvature. These features where combined with knowledge of the underlying image structure and the human anatomy to determine the quality of an isosurface. Although this method produces promising results for medical datasets, it is dependent on knowledge of the underlying object. In contrast to this, our method can quantify and visualize uncertainty independently from the underlying object and use this knowledge to optimize arbitrary geometry.

He et al. [8] presented an extension of the marching cubes algorithm, utilizing an uncertainty model to quantify uncertainty in image data. They transformed this information throughout the marching cubes algorithm. This approach leads to an uncertainty visualization complementing the extracted isosurface. Although this is a good starting point to introduce uncertainty information into an uncertainty-aware isosurface representation, the algorithm cannot indicate how the underlying uncertainty information affects the resulting position of geometric objects.

3 METHODS

We present a methodology to model and visualize uncertain geometry using multi-variate normal distributions. The general workflow is shown in Figure 1. Points, lines (line segments) and triangles can be modeled through 3D normal distributions (a) A user-defined grid is utilized to evaluate the uncertain geometry (b-d) Based on the evaluation grid, the geometry can be visualized (e) choosing different isovalues determining the minimally desired probability of a geometry. These steps are described in detail in the following section.

3.1 Modeling of Uncertain Geometry

Contrary to classical geometry, various applications deal with geometry that cannot be determined exactly. Therefore, an uncertainty-aware description of geometric objects is highly desirable. In the following, uncertainty-ware points are defined and their generalization to lines and triangles is explained.

Probabilistic Points Instead of defining fixed points or, higher-dimensional geometric objects, an uncertainty-aware description requires a function that is defined over the entire space $N(\mathbb{R}^3) \to \mathbb{R}$. This function can be evaluated at each point in space defining the probability density of an uncertainty-aware point to be located at a specific evaluated location. The function N satisfies all requirements of a probability density function. The most important is that $\int(N) = 1$. If N is a function returning 1, for a specific position in space, it describes the case of classic geometry.

We utilize a specific function type that fulfills this requirement: a multi-variate normal distribution [17]. This function type is a generalization of the Gaussian normal distribution to an n-dimensional setting. We limit the dimension to three as we tackle Euclidean geometry in this work. In general, the following description of an uncertainty-aware point can be extended to an arbitrary dimension without introducing further computational effort.

A three-dimensional uncertainty-ware point is defined as:

$$N_{\mu,\Sigma}(p) = \frac{1}{\sqrt{(2\pi)^3 det(\Sigma)}} e^{-\frac{1}{2}(p-\mu)^T \Sigma^{-1}(p-\mu)}, \tag{1}$$

where μ is the position of the normal distribution and Σ is the co-variance matrix. The input of the function is a point p. An example of a multi-variate normal distribution function and its input parameters is provided in Figure 1 a). The function returns the probability density for the uncertainty-ware point to be located at the specific location p. For each point in an uncertainty-aware geometry, the function $N_{\mu,\Sigma}(p)$ needs to be defined. The values of μ and σ are in general not equal for all points of an uncertain geometry. Based on this uncertain description of points, it is possible to define uncertain lines and triangles.

Probabilistic Geometry. Starting from uncertain points, modeled by multi-variate normal distributions, higher-dimensional simplices such as lines and triangles can be modeled. In classical geometry, lines, triangles and other simplices are based on points that can be connected. For the example of a line, this means that there are infinately many points directly connecting the end points of a line (line segment).

To model uncertain simplices this concept can be utilized as well. Therefore, uncertain simplices are formed by uncertain points that can be connected. For a line, this means that two uncertain points $A_{\mu_A,\Sigma_A}(p)$ and $B_{\mu_B,\Sigma_B}(p)$, and there are infinitely many points connecting A and B, which themselves are uncertain points. This set of points can be defined via linear interpolation of the points A and B, while interpolating the parameters μ and Σ of A and B.

The concept of linear interpolation can be extended to an arbitrary n-simplex [9]. Therefore, the presented concept can be utilized to model arbitrary uncertain n-simplex data without any further computational effort. Still, this paper focuses on the modeling of simplices up to dimension two (point, line segment, triangle) and their visualization.

The entirety of all uncertain points ($N_{N_{\mu_1,\Sigma_1}}(p)$), lines ($N_{N_1,N_2}(p)$) and triangles ($N_{N_1,N_2,N_3}(p)$) is referenced to as an uncertain geometry $G_U(p)$ in this manuscript.

3.2 Evaluation of Uncertain Geometries

In order to inspect and understand uncertain geometry, we present a methodology to evaluate uncertainty-aware geometry as well as a visualization for it. Contrary to classical geometry, this cannot be done in a straight-forward manner. The question is how to visualize the infinitely many multi-variate normal distributions defined through an uncertainty-aware geometry.

Our aim is to evaluate an uncertain geometry at regular points in space to identify the probability for each evaluation point; thus, our goal is to define whether the geometry of a point is actually located at a grid point. The size and position of this grid can be defined by the user.

A 2D example is shown in Figure 1 b). For each of the grid points, the uncertain geometry needs to be evaluated to identify the probability that the geometry is located at this grid point, i.e., that the geometry is present at the requested point. To achieve this, we need to evaluate the probability density of all points, lines and triangles to be present at the requested point and use the absolute maximum of all evaluated objects, see Figure 1 c) and d). The result of this computation is a regular grid storing the probability density for the evaluated geometry to be located at each grid point.

The evaluation of uncertainty-aware points was discussed in Section 3.1. In the case of a line (triangle), we are able to evaluate points and their probabilities as long as they are located on the line (in the triangle) itself. For points not located on the geometric object itself, we need to identify a point on the line (in the triangle) that can be evaluated vicariously for the entire geometric object. We use the perpendicular point of each line (triangle) to identify the point of the object with the highest influence on the evaluation point.

Overall, the probability of an uncertain geometry G_U is the highest evaluation response of all geometric objects contained in the geometry. Therefore, to calculate the influence of the entire geometry G_U for an evaluation point of the grid (p), we compute the value of the function $G(p)$, i.e.,

$$G_U(p) = max(\overbrace{N_{N_{\mu_1,\Sigma_1}(p)}}^{\text{Points}} \cup \overbrace{(N_{N_1,N_2}(p))}^{\text{Lines}} \cup \overbrace{(N_{N_1,N_2,N_3}(p))}^{\text{Triangles}}), \qquad (2)$$

where

- $N_{N_{\mu_1,\Sigma_1}}(p)$ are the evaluation functions of all points in the given uncertain geometry G_U;
- $N_{N_1,N_2}(p)$ are the evaluation functions of all lines in the given uncertain geometry G_U, and the function computes the perpendicular point of the evaluation point to the line and evaluates this point; and
- $N_{N_1,N_2,N_3}(p)$ are the evaluation functions of all points in the given geometry G_U, where the function computes the perpendicular point of the evaluation point to the triangle and uses this point for evaluation.

Although this computation is an exact evaluation of uncertain geometry, an evaluation of all geometric objects in an geometry for each of the evaluation points would be computationally expensive. Therefore, the computation of $G_U(p)$ needs to be approximated to achieve a lower computational complexity for the evaluation of uncertain geometry.

Let n be the number of of points, l be the number of lines and t be the number of triangles in an uncertain geometry. Furthermore, let m be the number of grid points that need to be evaluated. The resulting complexity for the evaluation of all geometric objects in each grid point would be $O(n*l*t*m)$, when considering the evaluation of each geometric object as a constant operation. For large grids and complex geometries, this complexity is too high.

To solve this problem, our method does not evaluate the entire geometry for each grid point. Instead, users can define a search radius that determines the geometric objects that are evaluated for the respective evaluation grid point. To achieve this, the algorithm requires two data structures. First, an octree is required, that is possible to search points that are located closely to the grid point and therefore minimize the number of objects that need to be evaluated. A search in the octree can be accomplished in $O(log(n))$-time. Second, for each uncertainty-aware point, the presented algorithm stores all lines and triangles the current point is part of. This approach supports fast access to lines and triangles located in the search radius of a grid point ($O(1)$). Using these two improvements, the algorithm is able to evaluate an uncertainty-aware geometry on a grid with m points in a complexity of $O(log(n)*m)$-time which is a significant improvement in complexity.

The result is an evaluation grid that holds the probability for the underlying uncertain geometry to be located at specific grid points. Based on this evaluation grid, uncertain geometry can be visualized in an intuitive manner.

3.3 Visualization of Uncertain Geometry

Based on the evaluation grid, uncertain geometry can be visualized to allow domain scientists to understand their data and the associated uncertainty. Therefore, mainly two concepts can be used to visualize the evaluated grid: volume rendering or isosurface extraction. Isosurfaces are able to visualize clear borders, which are helpful in indicating areas where a geometry can be located. In addition, isosurfaces have a lower amount of visual clutter, as they are not visualizing the entire space but only boundaries of regions. Furthermore, thin structures are hard to visualize through volume rendering, but they can clearly be indicated through isosurfaces. At last, isosurfaces are an intuitive choice for geometric objects.

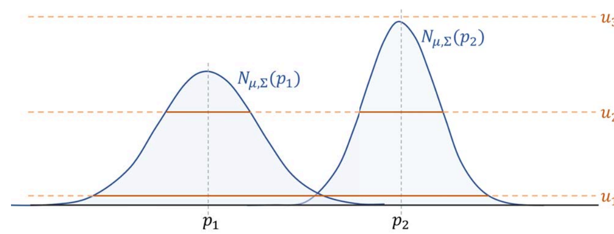

Figure 2: One dimensional example of two uncertainty-aware points. The probability density of each points outputs different results for the U-surface according to the selected treshold.

The presented visualization consists of two isosurfaces: the μ-surface (gray) and the U-surface (blue), shown in Figure 3.

The μ-surface is a surface representation of the underlying geometry considering all μ values of the multi-variate normal distributions and their connections to lines and triangles. Solely considering this surface would result in a standard visualization of a classical surface.

In addition to the μ-surface, the uncertainty information of the underlying geometry can be visualized by an additional surface, called U-surface. This surface is generated through an isosurface extraction based on the evaluation grid. The user can determine an arbitrary isovalue $u \in max(G_U)$. Based on this value, the U-surface can be generated, as shown in the one-dimensional example of Figure 2. The value indicates the minimal probability that is required for uncertain geometry to be present at an arbitrary point of the evaluation grid. This surface indicates the positional uncertainty of the underlying geometry. It is a surrounding hull of the μ-surface. The closer this surface is located to the μ-surface, the lower the positional uncertainty. When the surface is further away from the μ-surface, a higher degree positional uncertainty of geometry is implied. If no $U - surface$ is visible, this means, that there is not positional uncertainty when considering the user-defined threshold.

Figure 2 shows how the selection of the parameter u changes the resulting U-surface output. It ranges from no response (u_3) over non continuous response (u_2) to a continuous response (u_1).

In summary, we have presented a methodology that can model uncertain geometry by using multi-variate normal distributions to indicate locational uncertainty of the geometry.

4 Results and Discussion

The following Section will present visualizations origin from our methodology and discusses these results.

4.1 Results

Based on the presented methodology we have visualized two geometrical objects. The datasets originate from 3D image data, where interesting structures where extracted by a marching cubes algorithm [1].

Example 1: Aneurysm Usually, in clinical daily routine, CT scans are reviewed by using a slice-by-slice rendering technique. Unfortunately, with this technique it can be hard to follow the very thin vessels of an aneurysm. Due to this problem, a geometric description of the aneurysm is desired. As image data is affected by reconstruction errors introducing uncertainty, a direct isosurface extraction

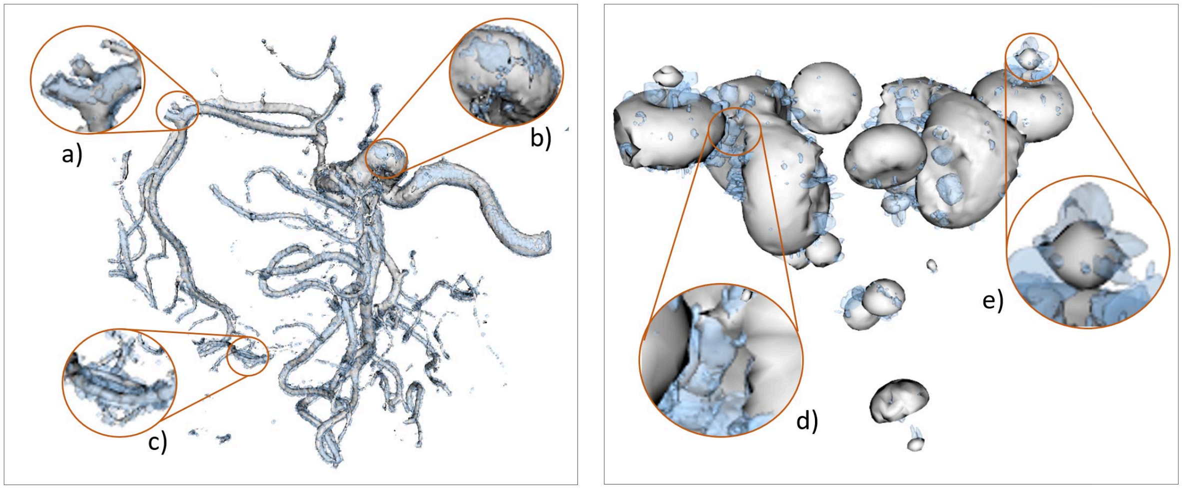

Figure 3: Resulting visualizations based on the presented methodology. a)-c) uncertainty-aware visualization of an aneurysm geometry. d)-e) uncertainty-aware visualization of an fluid phase surface.

might be slightly incorrect and does not communicate uncertainty information resulting in a potentially misleading representation. For the original image data, a set of uncertainty measures were used. See [6] for information regarding the uncertainty measures. These uncertainty measures where utilized to determine the Σ values of the multi-variate gaussian distribution.

Figure 3 shows the extracted surface from a CT scan capturing an aneurysm visualized with the presented method. The original image data has a size of 256x256x256 voxels, whereas the chosen evaluation grid has a size of 300x300x300 voxels. The chosen isovalue for the uncertainty surface was set to 0.001, which means that all possible geometries are covered that have a probability density larger than 0.1%. Figure 3 shows how the geometry can alter its position when considering the uncertainty information. a) shows a close-up of a large branch in the original geometry. The uncertainty surface shows that this size can vary strongly considering the available uncertainty information. b) shows a close-up of a big branch located at the origin of the aneurysm. Here, the position of the aneurysm does not change significantly and, therefore, one can be sure that the aneurysm is almost of that shape. c) shows a close-up of a very small side branch. The uncertainty surface changes considerably from the original surface. In addition, the surface can also be smaller than the visualized geometry, which can be viewed via a cutting plane. Therefore, the uncertainty surface indicates that this specific vessel can be very thin or relatively big, compared to other vessels in the aneurysm.

The presented method is very promising as it could assist medical doctors to communicate risks in surgeries and discuss different options for treatment while considering different configurations in the aneurysm's geometry.

Example 2: Fluid Simulation The second example is the geometry extracted from a fluid simulation data set. In the given CT scan, two phases are visible. We generated 100 slightly varying geometries to simulate a geometric ensemble. To generate Σ for each multivariate normal distribution, we utilized a fitting algorithm [10, 14]. The original image data has a resolution of 256x256x256, the utilized evaluation grid has a resolution of 300x300x300.

Figure 3 d) and e) shows the resulting geometry of the fluid interface and the uncertainty surface generated by the presented technique. The image shows that the U-Surface and the μ-surface show high differences, when the underlying ensemble data shows a lot of variety.

With our visualization method, users obtain an impression of how μ can change according the uncertainty that affects the geometric data.

4.2 Discussion

The presented methods requires the user to define a grid, a search radius and the parameter u. The grid should be at least as big as the region of interest for the user. The evaluation points should be chosen, thus to cover the double resolution of the $\mu - Surface$. For the search radius, the ten closest points outputted meaningful results in the presented cases. The choice of u is highly depending on the application.

In general, using the presented type of evaluation for uncertain geometry supports a high degree of freedom. One advantage is that the probability functions for geometric objects can be replaced easily. Furthermore, this method does not require a geometric construction to obtain the U-surface, which eliminates geometric or topological inconsistencies. The resulting visualizations are intuitive, as the U-surface indicates the positional uncertainty of the underlying μ-surface. Furthermore, the underlying concept of modeling uncertain geometry can be extended to higher dimensions.

5 Conclusion

We have presented a novel visualization methodology to visualize geometric data that contains uncertainty information by using multi-variate normal distributions. The resulting uncertainty-aware geometry description can be evaluated and visualized via isosurfaces. The involved computation is fast, and we avoid the problem of self-intersections. Our method provides an intuitive means for understanding geometry with uncertainty via a flexible and easy-to-use visualization.

As a future goal, we plan to devise a generalization of the underlying geometric description for defining probabilistic functions.

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