Hierarchical Morse-Smale Complex in 3D

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ABSTRACT

This paper investigates the topology of piecewise linear scalar functions, such as the interpolation function over a tetrahedralized volumetric grid. An algorithm is presented that creates a hierarchical representation of a Morse-Smale complex based on persistence. We show the correctness of our approach and discuss the applications.

1. INTRODUCTION

Traditional methods for exploring volumetric data involve the extraction of isosurfaces, or the visualization of some surface property based on a transfer function. An alternative to visualizing level set information is to visualize topological information. The *Morse-Smale complex* is a structure that partitions a dataset into topologically distinct regions based on gradient flow. Such a complex has many uses: a visualization of the complex can represent effectively the topology of a dataset; a hierarchical view of the topology is useful in topological smoothing, which can be applied in several areas, including medical imaging.

One of the problems with the construction of a Morse-Smale complex is the complexity inherent in dealing with the geometry of a 3-manifold. The Morse-Smale complex is well-defined over *Morse functions* as the intersection of stable and unstable manifolds. A Morse function is a smooth function where all the critical points are non-degenerate. However, a scalar dataset does not necessarily conform to the requirements of Morse functions. Therefore, for the desired properties of Morse theory to hold, such as error guarantees, the complex must be constructed from a given discrete scalar dataset in such a way that it implies a smooth Morse function.

We show how to construct a multi-resolution hierarchy given a valid Morse-Smale complex. This hierarchy helps us to resolve many of the questions regarding the validity of a derived complex, and therefore aids in the construction of the complex. By understanding how a complex can be simplified, we can perceive what a minimal configuration is as well as observe the valid connectivity of the complex. We present an algorithm for creating a multi-resolution hierarchy from a valid Morse-Smale complex, and discuss the applications of this hierarchy.

2. BACKGROUND

Some relevant terminology is introduced very briefly here.

A ***Morse function*** is a smooth mapping from an *N-manifold* to ***R***, the set of real numbers.

Critical Points of a Morse function are points where the gradient is zero. These include *maxima, minima,* and *saddle* points, see Figure 1.

Integral Lines are lines that start at a regular point, and follow the gradient vector field to a *maximum* or *minimum*. Integral lines starting at saddle points trace out some of the edges of the Morse-Smale complex.

Stable/Unstable Manifolds are the regions surrounding extrema, such that all integral lines starting inside the region end at the extrema. The *stable manifold* refers to the region associated with a *minimum*, and the *unstable manifold* refers to the region associated with a *maximum*. The surfaces created by the intersection of these manifolds partition the dataset into topologically distinct regions. Figure 2 illustrates these for a simple dataset.

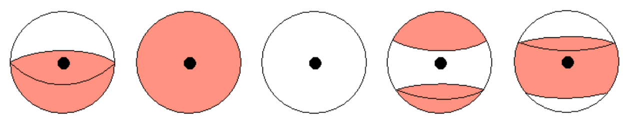
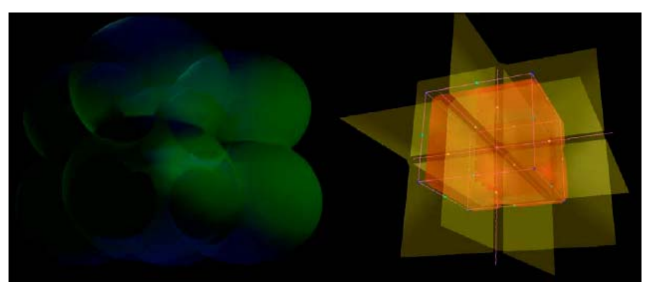


Figure 1 The types of critical points in a 3-manifold setting. The darker regions indicate lower function value compared to the value at the point in the middle, and the clear regions indicate higher function value. From left to right we have *regular, maximum, minimum, 1-saddle,* and *2-saddle*.



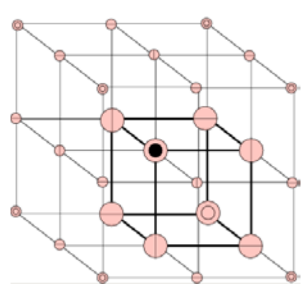


Figure 2 A simple dataset of distances to points arranged in a cube.

Top left shows the isosurface.

Top right shows the boundaries of the manifolds. The yellow sheets show the boudary of the stable manifolds.

Left the corresponding Morse-Smale complex. A single cell is highlighted.

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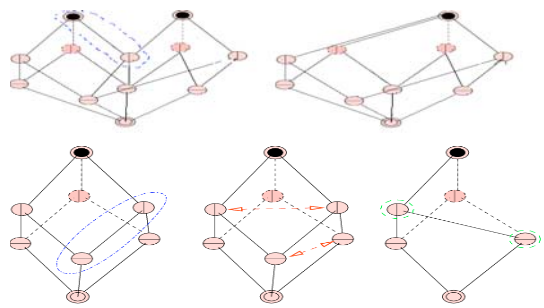

Figure 3 The top figure shows a 2-saddle - maximum contraction. Two cells are merged into one.

The bottom figure shows a 1-saddle - 2-saddle cancellation. An entire face of the cell is removed.

The ***Morse-Smale Complex*** is a structure that partitions the dataset into topologically distinct regions. A cell of the complex is defined as the intersection of the stable and unstable manifolds associated with a minimum and maximum. Therefore, each point inside the volume defined by the cell has the property that the integral line following the positive gradient ends at the maximum of the cell, and the integral line following the negative gradient ends at the minimum of the cell. The complex is stored as a set of cells. Each cell is composed of one minimum, a maximum, and a ring of alternating 1-saddles and 2-saddles.

3. PREVIOUS WORK

Bremer et al [1] implemented a Morse-Smale complex in the two-dimensional case, and used it to create a topological segmentation of datasets. Furthermore, they used the complex to create a multiresolution data structure for interactive viewing. Edelsbrunner et. al. [2] described an algorithm for constructing Morse-Smale complexes on three-dimensional tetrahedralized domains, based on finding the boundary surfaces between cells by growing them from saddles. The problem with these approaches is that they attempt to find implicit structures directly, and therefore run into combinatorial and numerical problems.

4. CANCELLATIONS

Given a valid Morse-Smale complex, it is possible to perform cancellation of critical point pairs such that the implied function is still smooth over a 3-manifold. The construction of a hierarchical representation relies on the ability to simplify the Morse-Smale complex. A cancellation of critical points involves "merging" them together spatially and in function value. One erases "bumps" in the function values by eliminating the critical points that define those bumps. Figure 3 shows how to perform cancellations.

A cancellation can also be seen as the contraction of an edge of the complex. There are two types of cancellations that lead to a valid complex. An edge contraction can occur either between a saddle and an extremum, or between two saddle points. The first case intuitively corresponds to removing a bump from the function. The extremum is removed, and the resulting function is smoothed. The second type of cancellation also smoothes the function, but in a different way. It corresponds to removing a face from a Morse cell, thereby simplifying the shape of the bump.

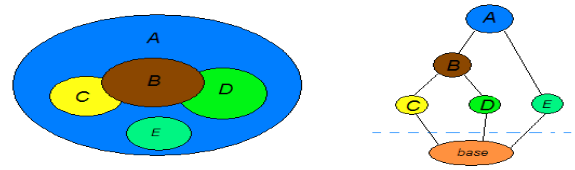

Figure 4 An abstract view of the space occupied by a complex. Node A represents the simplest complex. Each bubble corresponds to the region affected by a cancellation. Therefore, C, D, and E are independent cancellations, since their areas of influence do not overlap. On the right, the corresponding hierarchy is shown. In practice, the complex can be reconstructed at a desired resolution by traversing this hierarchy across a cut. Each node must store information on how to reconstruct the complex.

Certain cancellations are not allowed. The manifold structure of the function must be maintained, and the resulting complex must still be a Morse-Smale complex. Also, certain rules apply when determining how to reconnect the complex after a cancellation. Discussion of each case and test is beyond the scope of this short paper.

5. TOPOLOGICAL HIERARCHY

Applying successive cancellations leads to a topological hierarchy. The algorithm that simplifies a Morse-Smale complex sorts all pairs of critical points based on *persistence*, which is the absolute value of the difference of function value, and proceeds to simplify the complex in that order. This simplification order is important because it corresponds to removing the small features of the data before the large features; therefore, it is effective in removing noise. Furthermore, this order is important to maintain numerical correctness while making combinatorial decisions in cancellations. Failure to adhere to this order can lead to an ascending edge pointing from lower function value to higher function value.

Even though the ordering based on persistence is important, there is a set of *independent* cancellations at each point in the simplification process, each cancellation does affect the other cancellations in that set. Figure 4 illustrates the notion of independence and how a hierarchy is constructed.

6. RESULTS AND CONCLUSIONS

We developed a set of rules and an algorithm for performing simplification on a Morse-Smale complex. We have utilized the simplification process to construct a multi-resolution hierarchy for the complex.

7. REFERENCES

[1] P.-T. Bremer, H. Edelsbrunner, B. Hamann, and V. Pascucci, *A multi-resolution data structure for two-dimensional Morse functions,* Proceedings of IEEE Visualization, 2003, pages 139-146.

[2] H. Edelsbrunner, J. Harer, V. Natarajan, V. Pascucci. *Morse-Smale complexes for piecewise-linear 3-manifolds.*