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Curvature Approximation for Triangulated Surfaces

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Abstract. Given a set of points and normals on a surface and a triangulation associated with them a simple scheme for approximating the principal curvatures at these points is developed. The approximation is based on the fact that a surface can locally be represented as the graph of a bivariate function. Quadratic polynomials are used for this local approximation. The principal curvatures at a point on the graph of such a quadratic polynomial is used as the approximation of the principal curvatures at an original surface point.

Key words: Approximation, curvature, Gauss-Weingarten map, platelet, surface, triangulation.

1. Introduction

Methods for exactly calculating and approximating curvatures are important in geometric modeling for two reasons. In order to judge the quality of a surface one commonly computes curvatures for points on the surface, renders the surface's curvature as a texture map onto the surface and can thereby detect regions with undesired curvature behavior, such as surface regions locally changing from an elliptic to a hyperbolic shape. On the other hand, surface schemes are being developed requiring higher order geometric information as input, e.g., normal vectors and normal curvatures.

Definitions and theorems from classical differential geometry are reviewed as far as they are needed for the discussion. In classical differential geometry a surface is understood as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$,

$$\mathbf{x}(\mathbf{u}) = (x(u,v), y(u,v), z(u,v))^T \subset \mathbb{R}^3, \qquad \mathbf{u} \in D \subset \mathbb{R}^2. \tag{1}$$

The standard formulae are then used to derive techniques for approximating normal curvatures when a two-dimensional triangulation of a finite point set with associated outward unit normal vectors is given in three-dimensional space. Consequently, curvature estimates can be incorporated into existing surface generating schemes allowing curvature input. The quality of the curvature approximation is tested for triangulated surfaces obtained from a known parametric surface of the form $\mathbf{x}(\mathbf{u}) = (u, v, f(u,v))^T$.

Introductions to differential geometry are [Brauner '81], [do Carmo '76], [Lipschutz '80], [Strubecker '55, '58, '59], and [Struik '61]. Differential geometry is treated more analytically in [O'Neill '69]. One of the most comprehensive works on this subject is [Spivak '70]. Another reference in this field is [Farin '92]. Estimating Gaussian curvature from a discrete, triangulated point set is described in [Calladine '86]. Related triangle-based approximation and interpolation methods are discussed in [Akima '84], [Hagen & Pottmann '89], [Lawson '84], [Nielson & Franke '84], and [Renka & Cline '84]. Modelling triangulations arising in the context of contouring trivariate functions is treated in [Hamann '92].

2. Essential Terms of Differential Geometry

Some basic definitions of differential geometry are reviewed.

Definition 1. A **regular parametric two-dimensional surface** of class C^m $(m \geq 1)$ is the point set S in real three-dimensional space $\mathbb{R}^3$ defined by the mapping

$$\mathbf{x} = \mathbf{x}(\mathbf{u}) = (x(u, v), y(u, v), z(u, v))^T \tag{2}$$

of an open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^3$ such that all partial derivatives of x, y, and z of order m or less are continuous in U, and $\mathbf{x}_u \times \mathbf{x}_v \neq (0, 0, 0)^T$ for all $(u, v) \in U$.

Definition 2. The **tangent plane** at a point $\mathbf{x}_0 = \mathbf{x}(\mathbf{u}_0)$ on a regular parametric two-dimensional surface in three-dimensional space is defined as the set of all points $\mathbf{y}$ in $\mathbb{R}^3$ satisfying the equation

$$\mathbf{y} = \mathbf{x}_0 + a\mathbf{x}_u(\mathbf{u}_0) + b\mathbf{x}_v(\mathbf{u}_0), \qquad a, b \in \mathbb{R}. \tag{3}$$

Definition 3. The **outward unit normal vector** $\mathbf{n}_0 = \mathbf{n}(\mathbf{u}_0)$ of a regular parametric surface at a point $\mathbf{x}_0$ is given by

$$\mathbf{n}_0 = \frac{\mathbf{x}_u(\mathbf{u}_0) \times \mathbf{x}_v(\mathbf{u}_0)}{\|\mathbf{x}_u(\mathbf{u}_0) \times \mathbf{x}_v(\mathbf{u}_0)\|} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}, \tag{4}$$

where "$\| \ \|$" indicates the Euclidean norm.

Definition 4. Let $\mathbf{x}(\mathbf{u})$ be a regular parametric surface of class m, $m \geq 2$, and $\mathbf{c}(t) = \mathbf{c}(u(t), v(t))$ be a (regular) curve of class 2 on the surface through the point $\mathbf{x}_0 = \mathbf{x}(\mathbf{u}_0)$. The **normal curvature vector** to $\mathbf{c}(t)$ at $\mathbf{x}_0$ is the projection of the curvature vector $\mathbf{k} = \dot{\mathbf{t}}/\|\dot{\mathbf{t}}\|$, $\mathbf{t} = \dot{\mathbf{c}}/\|\dot{\mathbf{c}}\|$, onto the unit surface normal vector $\mathbf{n}_0$,

$$\mathbf{k}_n = (\mathbf{k} \cdot \mathbf{n}_0)\mathbf{n}_0 . \tag{5}$$

The proportionality factor $\mathbf{k} \cdot \mathbf{n}_0$ is called the **normal curvature**, denoted by κ_n.

Definition 5. The second degree polynomial

$$\begin{aligned} I(du, dv) &= \mathbf{x}_u \cdot \mathbf{x}_u \, du^2 + 2\mathbf{x}_u \cdot \mathbf{x}_v \, du \, dv + \mathbf{x}_v \cdot \mathbf{x}_v \, dv^2 \\ &= E \, du^2 + 2F \, du \, dv + G \, dv^2, \end{aligned} \tag{6}$$

where $du, dv \in \mathbb{R}$, is called the **first fundamental form** of a regular parametric surface $\mathbf{x}(\mathbf{u})$. The coefficients E, F, and G are called the **first fundamental coefficients**.

Definition 6. Assuming that the regular parametric surface $\mathbf{x}(\mathbf{u})$ is at least of order 2, the second degree polynomial

$$\begin{aligned} II(du, dv) &= -\mathbf{x}_u \cdot \mathbf{n}_u\, du^2 - (\mathbf{x}_u \cdot \mathbf{n}_v + \mathbf{x}_v \cdot \mathbf{n}_u)\, du\, dv - \mathbf{x}_v \cdot \mathbf{n}_v\, dv^2 \\ &= \mathbf{x}_{uu} \cdot \mathbf{n}\, du^2 + 2\mathbf{x}_{uv} \cdot \mathbf{n}\, du\, dv + \mathbf{x}_{vv} \cdot \mathbf{n}\, dv^2 \\ &= L\, du^2 + 2M\, du\, dv + N\, dv^2, \end{aligned} \tag{7}$$

where $du, dv \in \mathbb{R}$, is called the **second fundamental form** of $\mathbf{x}(\mathbf{u})$. The coefficients L, M, and N are called the **second fundamental coefficients.**

Definition 7. The two (real) eigenvalues κ_1 and κ_2 of the matrix

$$-A = -\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}, \tag{8}$$

where

$$a_{1,1} = \frac{MF - LG}{EG - F^2}, \qquad a_{1,2} = \frac{LF - ME}{EG - F^2},$$

$$a_{2,1} = \frac{NF - MG}{EG - F^2}, \qquad a_{2,2} = \frac{MF - NE}{EG - F^2},$$

of a regular surface of class of at least 2 at a point $\mathbf{x}_0$ are called **principal curvatures** of the regular parametric surface at $\mathbf{x}_0$. The associated eigenvectors determine the **principal curvature directions**. Therefore, the principal curvatures are the (real) roots of the characteristic polynomial of $-A$, the quadratic polynomial

$$\kappa^2 + (a_{1,1} + a_{2,2})\kappa + a_{1,1}a_{2,2} - a_{1,2}a_{2,1}. \tag{9}$$

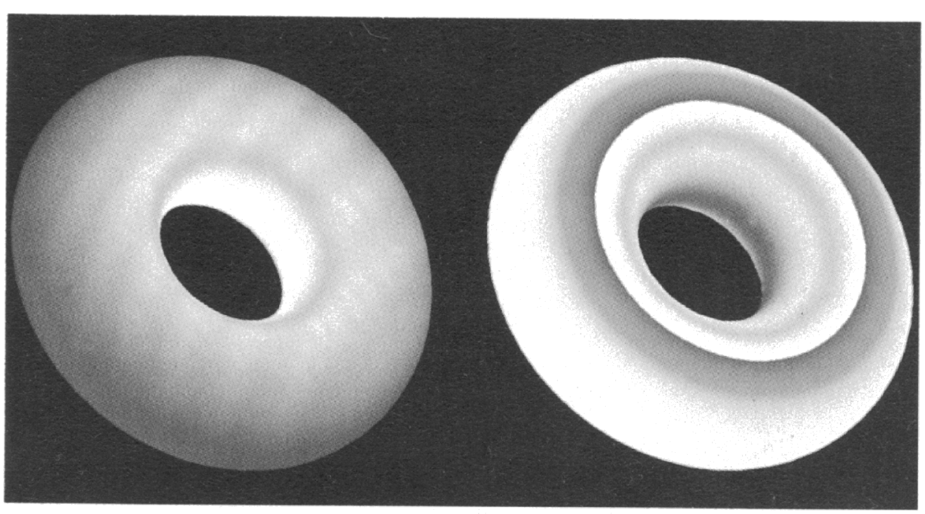

Figure 1. Texture map of mean and Gaussian curvature onto a torus, $((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u)^T$, $u, v \in [0, 2\pi]$; green/yellow representing negative curvature values, magenta/blue representing positive curvature values

Definition 8. The average H of the two principal curvatures κ_1 and κ_2 is called the **mean curvature**, the product K is called the **Gaussian curvature** of the regular parametric surface $\mathbf{x}(\mathbf{u})$ at $\mathbf{x}_0$,

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2. \tag{10}$$

3. Curvature Approximation for Triangulated two-Dimensional Surfaces

The graph of an explicit bivariate function $f(x, y)$ can be viewed as a special parametric surface with the parametrization $x(u, v) = u$, $y(u, v) = v$, and $z(u, v) = f(u, v)$,

$$\mathbf{x}(\mathbf{u}) = (u, v, f(u, v))^T, \qquad (u, v) \in D \subset \mathbb{R}^2, \tag{11}$$

The following formulae will be needed later on. Therefore, some basic facts are summarized next. For this particular surface, the unit normal vector is given by

$$\mathbf{n}(\mathbf{u}) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{(-f_u, -f_v, 1)^T}{\sqrt{1 + f_u^2 + f_v^2}}, \tag{12}$$

and the first and second fundamental coefficients are

$$E = 1 + f_u^2, \qquad F = f_u f_v \qquad G = 1 + f_v^2,$$

$$L = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}}, \quad M = \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}}, \quad \text{and} \quad N = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}}. \tag{13}$$

The Gauss-Weingarten map is

$$-A = -\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \frac{1}{l}\begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}\begin{pmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{pmatrix}^{-1}, \tag{14}$$

where $l = \sqrt{1 + f_u^2 + f_v^2}$.

Theorem 1. *Each regular parametric two-dimensional surface* $\mathbf{x}(\mathbf{u})$ *of class* m, $m \geq 2$, *can locally be represented in the explicit form* $z = z(x, y)$ *which is at least* C^2. *Choosing a surface point* $\mathbf{x}_0$ *as the origin of a local coordinate system and the z-axis in the same direction as the surface normal* $\mathbf{n}_0$ *at* $\mathbf{x}_0$, *z can be written as*

$$z(x, y) = \tfrac{1}{2}(c_{2,0}x^2 + 2c_{1,1}xy + c_{0,2}y^2) + \cdots, \tag{15}$$

Choosing appropriate basis vectors yields the representation of the **osculating paraboloid** *at* $\mathbf{x}_0$, *given by*

$$z(x, y) = \tfrac{1}{2}(c_{2,0}^* x^2 + c_{0,2}^* y^2),$$

such that the two principal curvatures at $\mathbf{x}_0$ *coincide with the coefficients of this paraboloid,* $\kappa_1 = c_{2,0}^*$ *and* $\kappa_2 = c_{0,2}^*$.

Proof. See [Strubecker '58, '59] or [Struik '61].

The principal curvature approximation method to be introduced is based on bivariate polynomials. It is essential to prove a certain property of such functions before describing the approximation technique. Given an origin in the plane, the graph of a bivariate polynomial f consisting of all the points in the set $\{(x, y, f(x, y))^T | x, y \in \mathbb{R}\}$ is independent of the choice of the orientation of the two unit vectors determining an orthonormal coordinate system for the plane. This fact implies that the principal curvatures of the graph, a two-dimensional surface, are independent of the two unit vectors as well.

Lemma 1. *The equation*

$$\sum_{k=0}^{i} (-1)^k \binom{i}{k} (x \cos^2 \alpha + y \sin \alpha \cos \alpha)^{i-k} (-x \sin^2 \alpha + y \sin \alpha \cos \alpha)^k = x^i \quad (16)$$

holds for all $x, y, \alpha \in \mathbb{R}$ and $i \geq 0$.

Proof. It is easy to show that Eq. (16) is valid for $i = 0$:

$$1 = x^0.$$

The induction hypothesis is made that Eq. (16) is true for $i - 1$. Thereby one proves that

$$\sum_{k=0}^{i} (-1)^k \binom{i}{k} (x \cos^2 \alpha + y \sin \alpha \cos \alpha)^{i-k} (-x \sin^2 \alpha + y \sin \alpha \cos \alpha)^k$$

$$= ((x \cos^2 \alpha + y \sin \alpha \cos \alpha) - (-x \sin^2 \alpha + y \sin \alpha \cos \alpha))$$

$$\sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} (x \cos^2 \alpha + y \sin \alpha \cos \alpha)^{i-1-k} (-x \sin^2 \alpha + y \sin \alpha \cos \alpha)^k$$

$$= x(\cos^2 \alpha + \sin^2 \alpha) x^{i-1} = x x^{i-1} = x^i.$$

□

Lemma 2. *The equation*

$$\sum_{l=0}^{j} \binom{j}{l} (x \sin \alpha \cos \alpha + y \sin^2 \alpha)^{j-l} (-x \sin \alpha \cos \alpha + y \cos^2 \alpha)^l = y^j \quad (17)$$

holds for all $x, y, \alpha \in \mathbb{R}$ and $j \geq 0$.

Proof. Follows the proof of lemma 1.

Theorem 2. *Let f be the bivariate polynomial*

$$f(x, y) = \sum_{\substack{i+j \leq n \\ i,j \geq 0}} c_{i,j} x^i y^j, \quad (18)$$

where a point in the plane has coordinates x and y with respect to a coordinate system given by an origin **o** *and two orthonormal basis vectors* $\mathbf{d}_1$ *and* $\mathbf{d}_2$; *rotating* $\mathbf{d}_1$ *and* $\mathbf{d}_2$ *around the origin* **o** *changes the representation of the bivariate polynomial, but not its graph.*

Proof. Let $\mathbf{d}_1$ and $\mathbf{d}_2$ be two unit vectors determining a first orthonormal coordinate system together with the origin **o**, and let $\bar{\mathbf{d}}_1$ and $\bar{\mathbf{d}}_2$ be a second pair of unit vectors obtained by rotating $\mathbf{d}_1$ and $\mathbf{d}_2$ by an angle α around **o**. A point in the plane may have coordinates $(x, y)^T$ with respect to the first coordinate system and coordinates

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \tag{19}$$

with respect to the second coordinate system. Assuming (18) is the representation of the polynomial f with respect to the first coordinate system, f can be rewritten using the inverse map of (19):

$$\begin{aligned} & f(x = \bar{x}\cos\alpha - \bar{y}\sin\alpha, y = \bar{x}\sin\alpha + \bar{y}\cos\alpha) \\ &= \sum_{\substack{i+j\le n\\ i,j\ge 0}} c_{i,j}(\bar{x}\cos\alpha - \bar{y}\sin\alpha)^i(\bar{x}\sin\alpha + \bar{y}\cos\alpha)^j. \end{aligned} \tag{20}$$

Evaluating f at the point $(\bar{x}, \bar{y})^T = (x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha)^T$, considering the binomial theorem, lemma 1, and lemma 2, one derives the equation

$$\begin{aligned} & f(\bar{x} = x\cos\alpha + y\sin\alpha, \bar{y} = -x\sin\alpha + y\cos\alpha) \\ &= \sum_{\substack{i+j\le n\\ i,j\ge 0}} c_{i,j}(\cos\alpha(x\cos\alpha + y\sin\alpha) - \sin\alpha(-x\sin\alpha + y\cos\alpha))^i \\ &\qquad (\sin\alpha(x\cos\alpha + y\sin\alpha) + \cos\alpha(-x\sin\alpha + y\cos\alpha))^j \\ &= \sum_{\substack{i+j\le n\\ i,j\ge 0}} c_{i,j}\left(\sum_{k=0}^{i}(-1)^k\binom{i}{k}(\cos\alpha(x\cos\alpha + y\sin\alpha))^{i-k}(\sin\alpha(-x\sin\alpha + y\cos\alpha))^k\right. \\ &\qquad \left.\sum_{l=0}^{j}\binom{j}{l}(\sin\alpha(x\cos\alpha + y\sin\alpha))^{j-l}(\cos\alpha(-x\sin\alpha + y\cos\alpha))^l\right) \\ &= \sum_{\substack{i+j\le n\\ i,j\ge 0}} c_{i,j}\left(\sum_{k=0}^{i}(-1)^k\binom{i}{k}(x\cos^2\alpha + y\sin\alpha\cos\alpha)^{i-k}(-x\sin^2\alpha + y\sin\alpha\cos\alpha)^k\right. \\ &\qquad \left.\sum_{l=0}^{j}\binom{j}{l}(x\sin\alpha\cos\alpha + y\sin^2\alpha)^{j-l}(-x\sin\alpha\cos\alpha + y\cos^2\alpha)^l\right) \\ &= \sum_{\substack{i+j\le n\\ i,j\ge 0}} c_{i,j}x^iy^j = f(x, y). \qquad \square \end{aligned}$$

The curvature approximation method is based on a localization of a two-dimensional triangulation. The local neighborhood around a point $\mathbf{x}_i$ is its platelet.

Definition 9. Given a two-dimensional triangulation in two- or three-dimensional space, the **platelet** $\mathscr{P}_i$ associated with a point $\mathbf{x}_i$ in the triangulation is the set of all triangles (determined by the index-triples (j_1, j_2, j_3) specifying their vertices) sharing $\mathbf{x}_i$ as a common vertex,

$$\mathscr{P}_i = \bigcup\{(j_1, j_2, j_3) | i = j_1 \vee i = j_2 \vee i = j_3\}. \tag{21}$$

The vertices constituting $\mathscr{P}_i$ are referred to as **platelet points**.

In order to approximate the principal curvatures at a point $\mathbf{x}_i$ in a two-dimensional triangulation a bivariate polynomial is constructed for a certain neighborhood around this point. Considering the facts that a two-dimensional surface can locally be represented explicitly (theorem 1) and that the graph of a bivariate polynomial is independent of the orientation of the two unit vectors determining an orthonormal coordinate system for the plane (theorem 2), the following sequence of computations is proposed.

(i) Determine the platelet points associated with $\mathbf{x}_i$.
(ii) Compute the plane P passing through $\mathbf{x}_i$ and having $\mathbf{n}_i$ (exact or approximated normal at $\mathbf{x}_i$) as its normal.
(iii) Define an orthonormal coordinate system in P with $\mathbf{x}_i$ as its origin and two arbitrary unit vectors in P.
(iv) Compute the distances of all platelet points from the plane P.
(v) Project all platelet points onto the plane, P and represent their projections with respect to the local coordinate system in P.
(vi) Interpret the projections in P as abscissae values and the distances of the original platelet points from P as ordinate values.
(vii) Construct a bivariate polynomial f approximating these ordinate values.
(viii) Compute the principal curvatures of f's graph at $\mathbf{x}_i$.

Let $\{\mathbf{y}_j = (x_j, y_j, z_j)^T \mid j = 0 \dots n_i\}$ be the set of all platelet points associated with the point $\mathbf{x}_i$ such that $\mathbf{y}_0 = \mathbf{x}_i$, and let $\mathbf{n} = (n^x, n^y, n^z)^T$ be the outward unit normal vector at $\mathbf{y}_0$. The implicit equation for the plane P is then given by

$$\begin{aligned}\mathbf{n}\cdot(\mathbf{x} - \mathbf{y}_0) &= n^x(x - x_0) + n^y(y - y_0) + n^z(z - z_0)\\ &= n^x x + n^y y + n^z z - (n^x x_0 + n^y y_0 + n^z z_0)\\ &= Ax + By + Cz + D = 0. \end{aligned} \tag{22}$$

Depending on the outward unit normal vector $\mathbf{n}$ one chooses a vector $\mathbf{a}$ perpendicular to $\mathbf{n}$ ($\mathbf{a}\cdot\mathbf{n} = 0$) among the possibilities

$$\mathbf{a} = \begin{cases} \dfrac{1}{n^x}(-(n^y + n^z), n^x, n^x)^T, & n^x \neq 0,\\[2ex] \dfrac{1}{n^y}(n^y, -(n^x + n^z), n^y)^T, & n^y \neq 0,\\[2ex] \dfrac{1}{n^z}(n^z, n^z, -(n^x + n^y))^T, & n^z \neq 0, \end{cases}$$

in order to obtain the first unit basis vector $\mathbf{b}_1$,

$$\mathbf{b}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|}, \qquad \|\mathbf{a}\| = \sqrt{(\mathbf{a}\cdot\mathbf{a})}\,.$$

The second unit basis vector $\mathbf{b}_2$ is defined as the cross product of $\mathbf{n}$ and $\mathbf{b}_1$,

$$\mathbf{b}_2 = \mathbf{n} \times \mathbf{b}_1.$$

The perpendicular signed distances d_j, $j = 0 \dots n_i$, of all platelet points $\mathbf{y}_j$ from the plane P are

$$d_j = \text{dist}(\mathbf{y}_j, P) = \frac{Ax_j + By_j + Cz_j + D}{\sqrt{A^2 + B^2 + C^2}} = Ax_j + By_j + Cz_j + D. \tag{23}$$

Projecting all platelet points $\mathbf{y}_j$ onto P yields the points $\mathbf{y}_j^P$,

$$\mathbf{y}_j^P = \mathbf{y}_j - d_j\mathbf{n}. \tag{24}$$

Considering $\mathbf{y}_0$ as the origin and $\mathbf{b}_1$ and $\mathbf{b}_2$ as the two unit basis vectors of a local two-dimensional orthonormal coordinate system for the plane P, each point $\mathbf{y}_j^P$ in P can be expressed in terms of that coordinate system. Therefore, one computes the difference vectors

$$\mathbf{d}_j = \mathbf{y}_j^P - \mathbf{y}_0, \qquad j = 0 \ldots n_i,$$

and expresses them as linear combinations of the two unit basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ in P. Each difference vector $\mathbf{d}_j$ can be represented in the form

$$\mathbf{d}_j = (\mathbf{d}_j \cdot \mathbf{b}_1)\mathbf{b}_1 + (\mathbf{d}_j \cdot \mathbf{b}_2)\mathbf{b}_2, \tag{25}$$

defining the local coordinates u_j and v_j of the point $\mathbf{y}_j^P$ in terms of the local coordinate system:

$$(u_j, v_j)^T = (\mathbf{d}_j \cdot \mathbf{b}_1, \mathbf{d}_j \cdot \mathbf{b}_2)^T. \tag{26}$$

Interpreting the local coordinates u_j and v_j as abscissae values and the signed distances d_j as ordinate values (in direction of the normal $\mathbf{n}$), a polynomial $f(u, v)$ of degree two (see theorem 1) is constructed approximating these ordinate values. Forcing the polynomial f to satisfy $f(0,0) = f_u(0,0) = f_v(0,0) = 0$, the constraints

$$f(u_j, v_j) = \tfrac{1}{2}(c_{2,0}u_j^2 + 2c_{1,1}u_jv_j + c_{0,2}v_j^2) = d_j, \qquad j = 1 \ldots n_i,$$

remain. Written in matrix representation these constraints are

$$\begin{bmatrix} u_1^2 & 2u_1v_1 & v_1^2 \\ \vdots & \vdots & \vdots \\ u_{n_i}^2 & 2u_{n_i}v_{n_i} & v_{n_i}^2 \end{bmatrix} \begin{bmatrix} c_{2,0} \\ c_{1,1} \\ c_{0,2} \end{bmatrix} = U\mathbf{c} = \mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_{n_i} \end{bmatrix}. \tag{27}$$

This overdetermined system of linear equations is solved using a least squares approach (see [Davis '75]). The resulting normal equations are

$$U^TU\mathbf{c} = U^T\mathbf{d}. \tag{28}$$

Provided the determinant of U^TU does not vanish this system can immediately be solved using Cramer's rule. In the case that the determinant of U^TU vanishes (e.g., when $\mathbf{x}_i$ is a point on the boundary of the triangulation) one considers additional points connected to $\mathbf{x}_i$'s platelet by an edge in the triangulation.

Theorem 3. *The principal curvatures κ_1 and κ_2 of the graph $(u, v, f(u, v))^T \subset \mathbb{R}^3$, u, $v \in \mathbb{R}$, of the bivariate polynomial*

$$f(u, v) = \tfrac{1}{2}(c_{2,0}u^2 + 2c_{1,1}uv + c_{0,2}v^2) \tag{29}$$

at the point $(0, 0, f(0,0))^T$ *are given by the two real roots of the quadratic equation*

$$\kappa^2 - (c_{2,0} + c_{0,2})\kappa + c_{2,0}c_{0,2} - c_{1,1}^2 = 0. \tag{30}$$

Proof. According to definition 7 and Eq. (14), the principal curvatures of f's graph are the eigenvalues of the matrix

$$-A = \frac{1}{l}\begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}\begin{pmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{pmatrix}^{-1},$$

where $l = \sqrt{1 + f_u^2 + f_v^2}$. Evaluating $-A$ for $u = v = 0$, one obtains the matrix

$$-A = \begin{pmatrix} c_{2,0} & c_{1,1} \\ c_{1,1} & c_{0,2} \end{pmatrix},$$

having the characteristic polynomial in (30). □

Solving the normal Eq. (28) and determining the roots of the characteristic polynomial in (30), one finally obtains the desired approximations for the principal curvatures at the point $\mathbf{x}_i$.

The above construction is illustrated in Fig. 2. Shown are the platelet points around the point $\mathbf{x}_i$, the tangent plane P, its local orthonormal coordinate system (origin $\mathbf{x}_i$ and basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$), and the projections of the platelet points ($\mathbf{y}_j^P$) onto P.

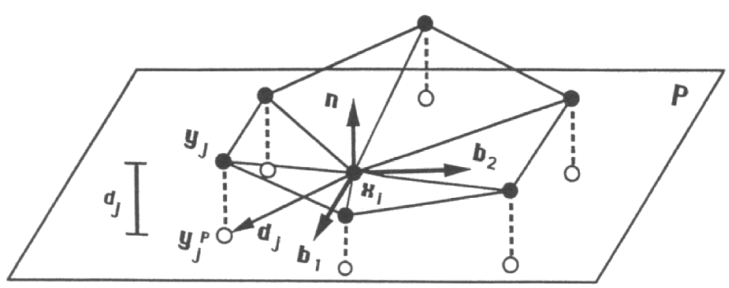

Figure 2. Construction of a bivariate polynomial for platelet points in a two-dimensional triangulation

4. Test Results

The presented technique for principal curvature approximation is tested for graphs of several bivariate functions. The exact principal curvatures κ_1^{ex} and κ_2^{ex} are compared with the approximated principal curvatures κ_1^{app} and κ_2^{app}; the exact mean curvature $H^{ex} = \frac{1}{2}(\kappa_1^{ex} + \kappa_2^{ex})$ is compared with the average of the approximated principal curvatures $H^{app} = \frac{1}{2}(\kappa_1^{app} + \kappa_2^{app})$ and the exact Gaussian curvature $K^{ex} = \kappa_1^{ex}\kappa_2^{ex}$ with the product of the approximated principal curvatures $K^{app} = \kappa_1^{app}\kappa_2^{app}$.

All bivariate test functions $f(x, y)$ are defined over $[-1, 1] \times [-1, 1]$ and evaluated on a $51 \cdot 51$—grid with equidistant spacing,

$$(x_i, y_j)^T = \left(-1 + \frac{i}{25}, -1 + \frac{j}{25}\right)^T, \qquad i,j = 0 \ldots 50,$$

determining a finite set of three-dimensional points on their graphs,

$$\{(x_i, y_j, f(x_i, y_j))^T \,|\, i,j = 0 \ldots 50\}.$$

The triangulation of a function's graph is obtained by splitting each quadrilateral specified by its index quadruple

$$((i,j), (i+1,j), (i+1,j+1), (i,j+1))$$

into the two triangles $T_{i,j}^1$ and $T_{i,j}^2$ identified by their index triples,

$$T_{i,j}^1 = ((i,j), (i+1,j), (i+1,j+1)) \quad \text{and} \quad T_{i,j}^2 = ((i,j), (i+1,j+1), (i,j+1)).$$

The root-mean-square error (RMS error) is a common error measure and is computed for each test example and curvature type. The RMS error is defined as

$$\sqrt{\frac{1}{n}\sum_{i=0}^{n-1} (f_i^{ex} - f_i^{app})^2}\,, \tag{31}$$

where n is the total number of exact (or approximated) values $f_i^{ex}(f_i^{app})$. Here, n equals $51 \cdot 51$; depending on the curvature type approximated f_i^{ex} can represent the exact values for κ_1^{ex}, κ_2^{ex}, H^{ex} or K^{ex}, and f_i^{app} can represent the approximated values for κ_1^{app}, κ_2^{app}, H^{app} or K^{app}, respectively. Table 1 summarizes the test results for the approximation of the principal curvatures, the mean, and the Gaussian curvature.

Table 1. RMS errors of curvature approximation for graphs of bivariate functions

Function	κ_1	κ_2	H	K
1. Cylinder: $\sqrt{2 - x^2}$.	.000291	.000035	.000132	.000025
2. Sphere: $\sqrt{4 - (x^2 + y^2)}$.	.000159	.000046	.000080	.000080
3. Paraboloid: $.4(x^2 + y^2)$.	.003073	.001342	.001358	.001684
4. Hyperboloid: $.4(x^2 - y^2)$.	.002058	.002058	.001057	.001767
5. Monkey saddle: $.2(x^3 - 3xy^2)$.	.004483	.004483	.001591	.007247
6. Cubic polynomial: $.15(x^3 + 2x^2y - xy + 2y^2)$.	.002258	.003598	.001665	.002242
7. Exponential function: $e^{-1/2(x^2+y^2)}$.	.001757	.005546	.002722	.002602
8. Trigonometric function: $.1(\cos(\pi x) + \cos(\pi y))$.	.002998	.002821	.001013	.003541

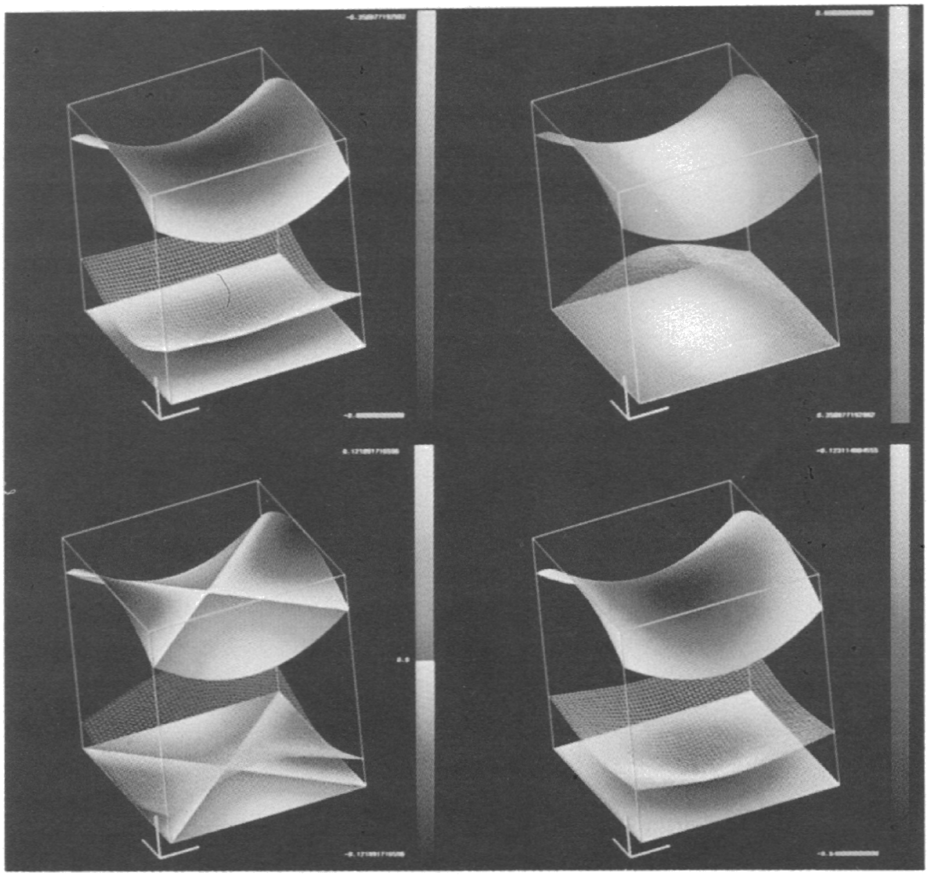

Figure 3. Exact curvatures κ_1^{ex}, κ_2^{ex}, H^{ex}, and K^{ex} on the graph of $f(x, y) = .4(x^2 - y^2)$, $x, y \in [-1, 1]$

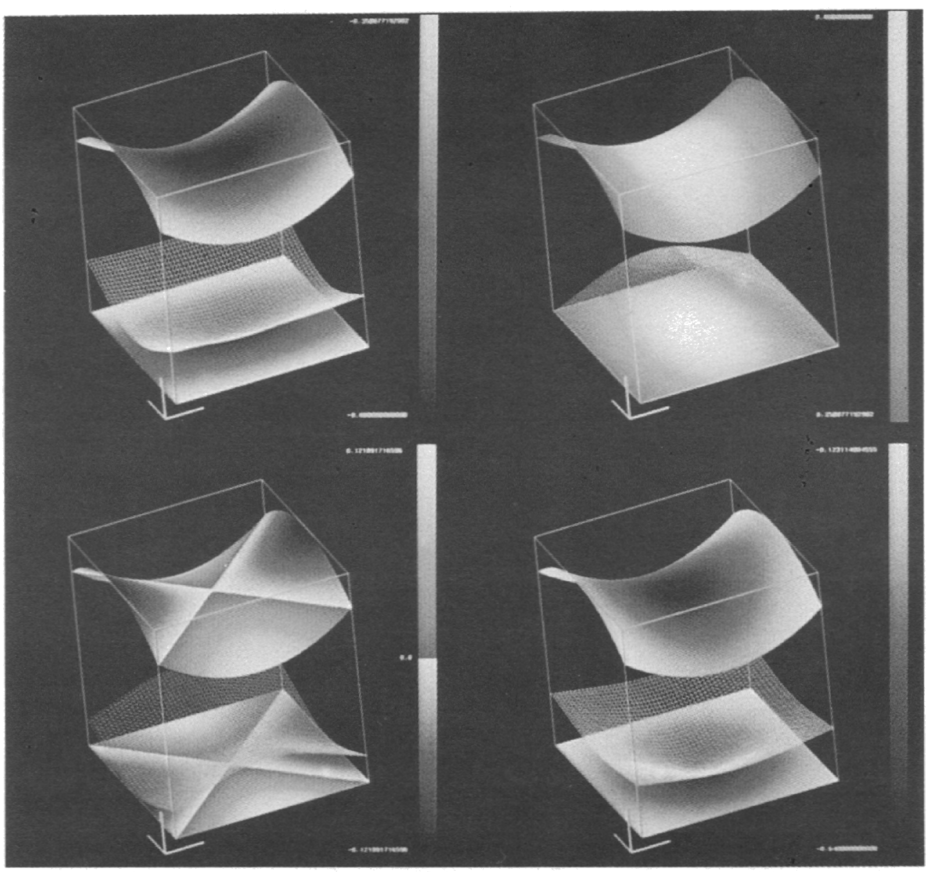

Figure 4. Approximated curvatures κ_1^{app}, κ_2^{app}, H^{app}, and K^{app} on the graph of $f(x, y) = .4(x^2 - y^2)$, x, $y \in [-1, 1]$

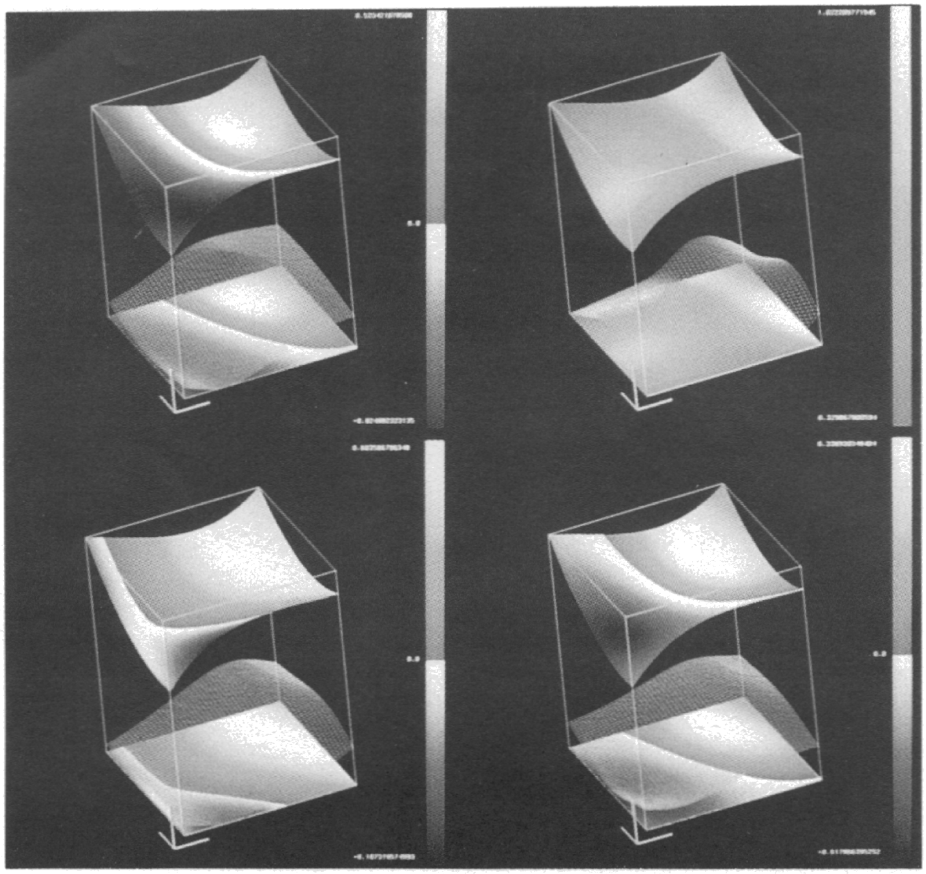

Figure 5. Exact curvatures κ_1^{ex}, κ_2^{ex}, H^{ex}, and K^{ex} on the graph of $f(x, y) = .15(x^3 + 2x^2y - xy + 2y^2)$, $x, y \in [-1, 1]$

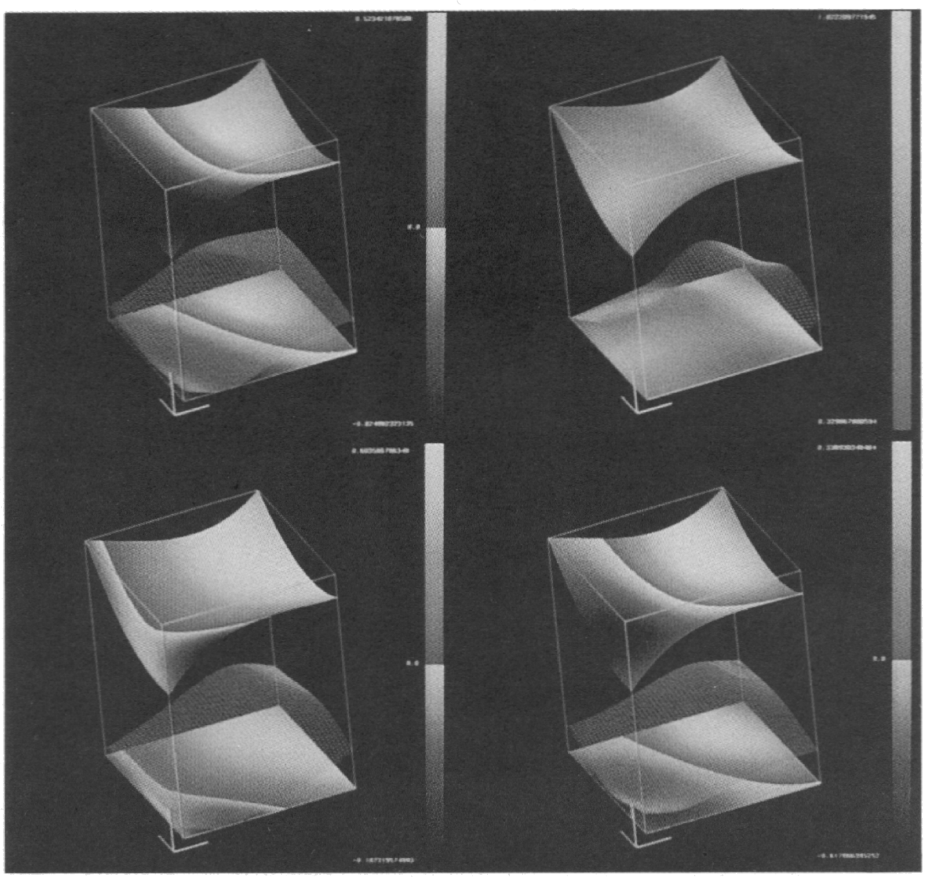

Figure 6. Approximated curvatures κ_1^{app}, κ_2^{app}, H^{app}, and K^{app} on the graph of $f(x, y) = .15(x^3 + 2x^2y - xy + 2y^2)$, $x, y \in [-1, 1]$

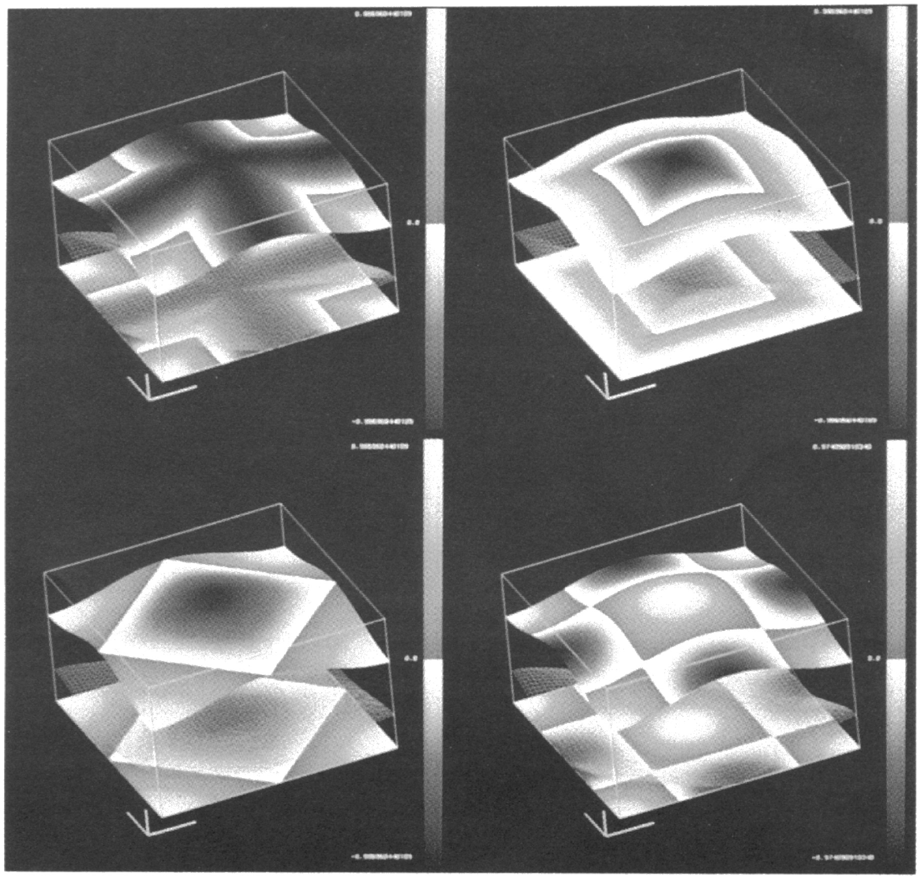

Figure 7. Exact curvatures κ_1^{ex}, κ_2^{ex}, H^{ex}, and K^{ex} on the graph of $f(x, y) = .1(\cos(\pi x) + \cos(\pi y))$, x, $y \in [-1, 1]$

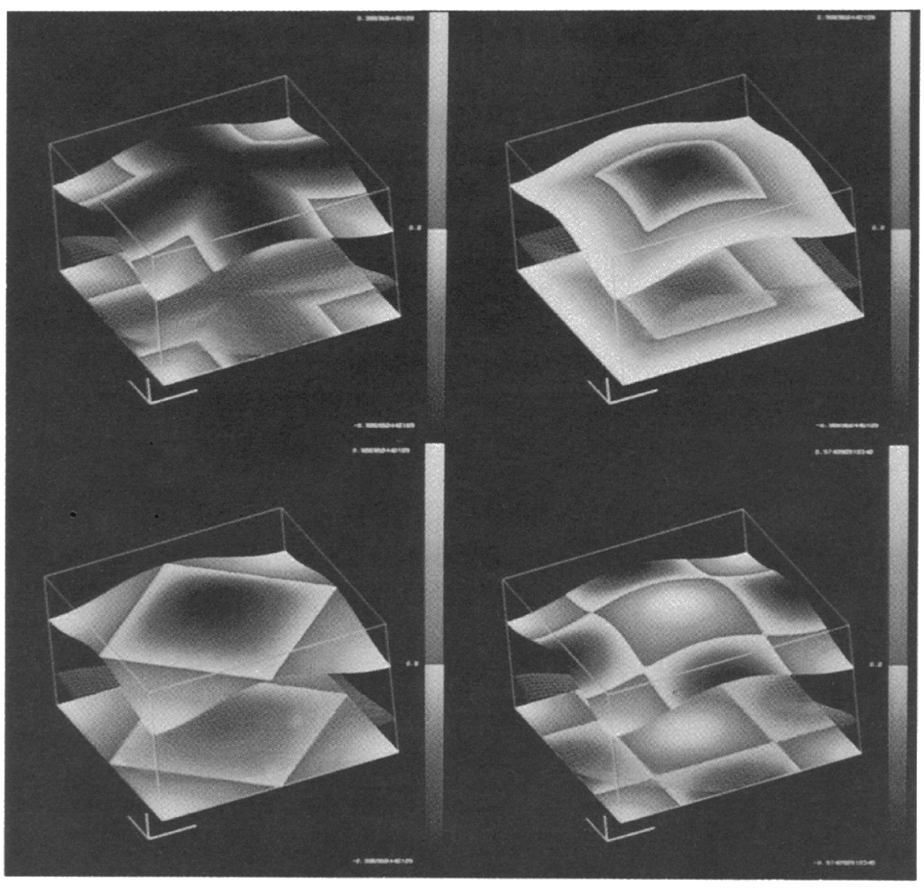

Figure 8. Approximated curvatures κ_1^{app}, κ_2^{app}, H^{app}, and K^{app} on the graph of $f(x, y) = .1(\cos(\pi x) + \cos(\pi y))$, x, $y \in [-1, 1]$

In the figures, the four particular curvatures used in Table 1 are mapped as textures onto the hyperboloid (function 5), the graph of the cubic polynomial (function 7) and the graph of the trigonometric function (function 9). Pairs of consecutive figures show the exact (upper figure) and the approximated curvatures (lower figure). The principal curvature κ_1 is visualized in the upper-left, κ_2 in the upper-right, the mean curvature H in the lower-left and the Gaussian curvature K in the lower-right corner of each figure. The Figs. 3 and 4 show the exact and approximated curvature values for function 5, the Figs. 5 and 6 for function 7, and the Figs. 7 and 8 for function 9.

5. Conclusions

A technique for approximating the two principal curvatures at the vertices in a two-dimensional surface triangulation has been developed. The test examples chosen are all graphs of bivariate functions leading to an obvious error measure. Nevertheless, the scheme should perform well for general surface triangulations, since all surfaces can locally be viewed as graphs of bivariate functions. At this point, it has not been investigated how to adjust the scheme to platelets which can not be described in terms of a function. One could use an implicit surface approximation whenever necessary.

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