Notes, Scales, Periodicity, Modular Arithmetic, Overtones and Overtone Series

Modular Arithmetic for the
Representation and Visualization of Notes and Scales as
Cyclic Matrices or Periodic Networks on a Torus and the
Relevance of Overtones

Relationships between
Basic Concepts of Music Theory and
Principles of Algebra, Arithmetic, Combinatorics,
Geometry, Graph Theory, Number Theory and Physics

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Sacramento, California

2026

Visualization of the Notes of a 12-tone System as an Effective Representation of Scales, Chords, Harmony and Their Relationships

6	7	8	9	10	11	12|0	1	2	3	4	5	6	
Gb	Db	Ab	Eb	Bb	F	C	G	D	A	E	B	F#	0|7
F	C	G	D	A	E	B	F#	C#	G#	D#	A#	E#	6
Fb|E	Cb|B	Gb|F#	Db|C#	Ab|G#	Eb|D#	Bb|A#	F|E#	C|B#	G|F##	D|C##	A|G##	E|D##	
Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#	D#	5
Ebb|D	Bbb|A	Fb|E	Cb|B	Gb|F#	Db|C#	Ab|G#	Eb|D#	Bb|A#	F|E#	C|B#	G|F##	D|C##	
Db	Ab	Eb	Bb	F	C	G	D	A	E	B	F#	C#	4
Dbb|C	Abb|G	Ebb|D	Bbb|A	Fb|E	Cb|B	Gb|F#	Db|C#	Ab|G#	Eb|D#	Bb|A#	F|E#	C|B#	
Cb	Gb	Db	Ab	Eb	Bb	F	C	G	D	A	E	B	3
Bb	F	C	G	D	A	E	B	F#	C#	G#	D#	A#	2
Bbb|A	Fb|E	Cb|B	Gb|F#	Db|C#	Ab|G#	Eb|D#	Bb|A#	F|E#	C|B#	G|F##	D|C##	A|G##	
Ab	Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#	1
Abb|G	Ebb|D	Bbb|A	Fb|E	Cb|B	Gb|F#	Db|C#	Ab|G#	Eb|D#	Bb|A#	F|E#	C|B#	G|F##	
Gb	Db	Ab	Eb	Bb	F	C	G	D	A	E	B	F#	0|7
F	C	G	D	A	E	B	F#	C#	G#	D#	A#	E#	6
Fb|E	Cb|B	Gb|F#	Db|C#	Ab|G#	Eb|D#	Bb|A#	F|E#	C|B#	G|F##	D|C##	A|G##	E|D##	
Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#	D#	5

This “matrix visualization” is a periodic row-and-column representation of 12 half-tones. Columns show vertically shifted 12-tone chromatic scales; rows show the horizontally shifted standard circle of fifths. E# = F, B# = C, Fb = E, Cb = B, F## = G, C## = D, G## = A, D## = E and Bbb = A, Ebb = D, Abb = G, Dbb = C.

The visualization above exhibits several interesting and relevant characteristics:

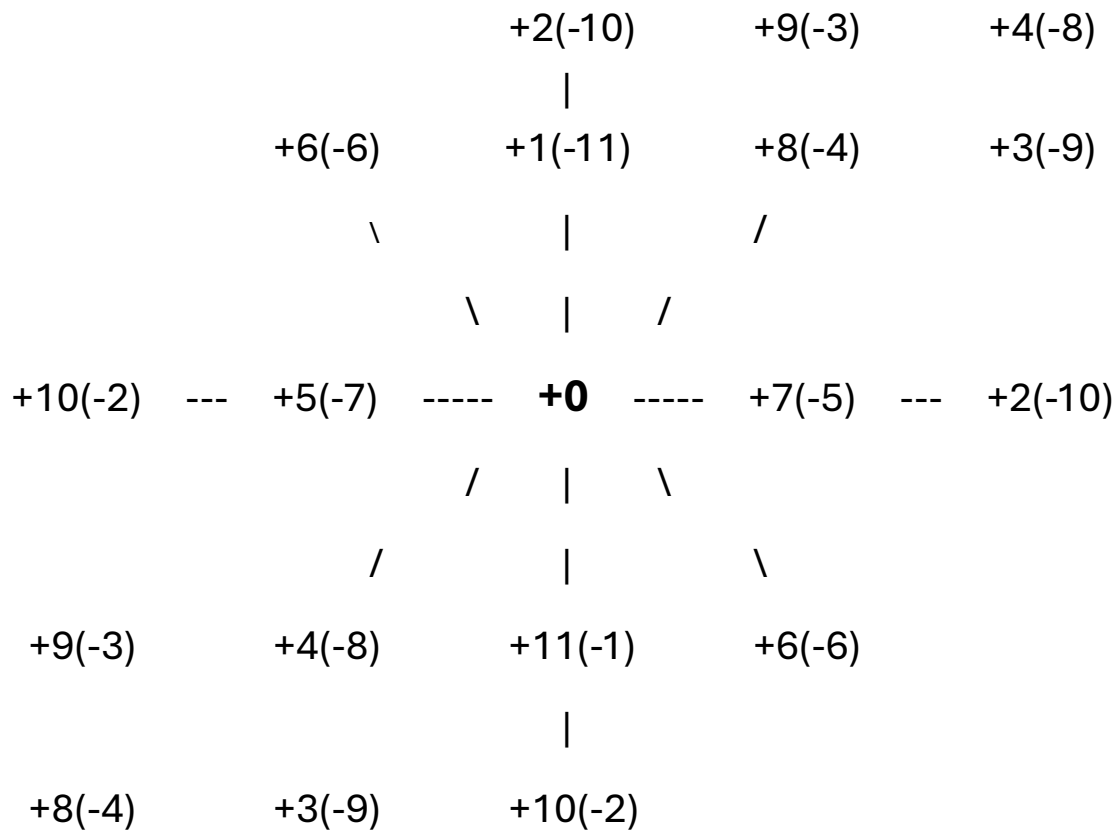
- The center column shows the 7(8) notes of the C-major and A-minor scales.
- The underlined 7-note neighborhood shows the 7 C-major (A-minor) notes, defining the 7 scale-specific major (3), minor (3) and diminished (1) triads of the C-major (A-minor) scale.
- In every row, the horizontal order of 12 notes represents the order of the basic circle of 5^{th}s.
- Horizontal rows in smaller font show notes between the notes in the row above and below.
- The numbered rows (columns) use modulo-8 (modulo-12) arithmetic, reflecting underlying periodicity of scales, defining a periodic network of notes/scales one can map onto a torus.

Gb	Db	Ab	Eb	Bb	F	C	G	D	A	E	B	F#
6	**1**	**8**	**3**	**10**	**5 F**	**0 C**	**7 G**	**2**	**9**	**4**	**11**	**6**
5	**0**	**7**	**2**	**9**	**4**	**1 B**	**6**	**1**	**8**	**3**	**10**	**5**
4	11	6	1	8	3	10	5	0	7	2	9	4
3	**10**	**5**	**0**	**7**	**2 D**	**9 A**	**4 E**	**11**	**6**	**1**	**8**	**3**
2	9	4	11	6	1	8	3	10	5	0	7	2
1	**8**	**3**	**10**	**5**	**0**	**7 G**	**2**	**9**	**4**	**11**	**6**	**1**
0	**7**	**2**	**9**	**4**	**11**	**6**	**1**	**8**	**3**	**10**	**5**	**0**
11	**6**	**1**	**8**	**3**	**10**	**5 F**	**0**	**7**	**2**	**9**	**4**	**11**
10	**5**	**0**	**7**	**2**	**9**	**4 E**	**11**	**6**	**1**	**8**	**3**	**10**
9	4	11	6	1	8	3	10	5	0	7	2	10
8	**3**	**10**	**5**	**0**	**7**	**2 D**	**9**	**4**	**11**	**6**	**1**	**8**
7	2	9	4	11	6	1	8	3	10	5	0	7
6	**1**	**8**	**3**	**10**	**5 F**	**0 C**	**7 G**	**2**	**9**	**4**	**11**	**6**
-7(+5)	-7(+5)	-7(+5)	-7(+5)	-7(+5)	-7(+5)	0[12]	+7(-5)	+7(-5)	+7(-5)	+7(-5)	+7(-5)	+7(-5)

Modulo-12 matrix using half-tone indexing with numbers in- and decreasing by 5 or 7 in rows and increasing by 1 in columns (from bottom to top), where (C,D,E,F,G,A,B,C) = (0,2,4,5,7,9,11,12|0). (The notation "L|R" indicates that 'L' and 'R' are equivalent values.)

The next two illustrations focus on a note's local neighborhood of this modulo-12 matrix.

Local Index Neighborhood around a Central Note (+0)



Example: Local Neighborhood around the Note C

					Implied Triads
		D	A	E	D-min, A-min, E-min
	Gb|F#	Db|C#	Ab|G#	Eb|D#	Half-tones between
(Bb)	F	**C**	G	(D)	F-maj, **C-maj**, G-maj
(A)	(E)	B	(Gb|F#)		B-diminished
(Ab|G#)	(Eb|D#)	Bb|A#			Half-tones between

The next modulo-12 representation uses a “12-hour clock visualization” for 12 half-tones.

	0	0	0	C
	1	0	11	B
	2	0	10	Bb|A#
	3	0	9	A
	4	0	8	Ab|G#
	5	0	7	G
	6	0	6	Gb|F#
	7	0	5	F
	8	0	4	E
	9	0	3	Eb|D#
	10	0	2	D
	11	0	1	Db|C#
	0	0	0	C

Chromatic Scale

0 2 4 6 8 10 0 2 4 6 8 10 0 → 0 2 4 6 8 10 0 2 4 6 8 10 0

0 3 6 9 0 3 6 9 0 3 6 9 0 0 3 6 9 0 3 6 9 0 3 6 9 0

0 4 8 0 4 8 0 4 8 0 4 8 0 ← 0 4 8 0 4 8 0 4 8 0 4 8 0

	Dbb|C	0	0	0
	Abb|G	7	6	5
	Ebb|D	2	0	10
	Bbb|A	9	6	3
	Fb|E	4	0	8
	Cb|B	11	6	1
	Gb|F#	6	0	6
	Db|C#	1	6	11
	Ab|G#	8	0	4
	Eb|D#	3	6	9
	Bb|A#	10	0	2
	F|E#	5	0	7
Circle of Fifths	C|B#	0	0	0

12-hour Clock Visualization Principle. This principle guides the visualization above. The sketch below shows the "12 directions of a clock," as defined by marks for hour 0, 1, 2, ..., 12|0 marks. In all 12 directions, scales are defined by moving 1 half-tone (+1 direction), 2 half-tones (+2 direction), 3 half-tones (+3 direction), ..., 11 half-tones (+11 direction) or 12 half-tones (+12 direction) in outward direction from the center of the clock. Half-tones are identified with their equivalent integer indices, using modulo-12 arithmetic. For example, C=0|12, C#=1, D=2, D#=3, E=4, F=5, F#=6, G=7, G#=8, A=9, A#=10, B=11. The figure above shows the resulting 12 scales one obtains when constructing scales based on such a clock visualization principle.

Sketch of the "12-hour clock"

	+11	+12|+0	+1		When moving in the +1 direction,
+10		→		+2	one obtains the chromatic scale
+9				+3	from C = 0|12 to B = 11. When moving
+8		←		+4	in the +7 direction, one obtains the
	+7	+6	+5		circle-of-fifths, i.e., C, G, D, ..., A#,

E#|F and B#|C. Alternatively, one can interpret the result of this construction principle as circles having "distances" 0, 1, 2, ..., and 12 from the center of the clock. Consequently, the chromatic scale from C to C (above) is defined by moving clockwise on the circle with distance 1 from the center of the clock, while the circle-of-fifths is defined by moving clockwise on the circle with distance 7 from the center of the clock. One can adopt a geometrical viewpoint and interpret the construction result as two families of circles, 12(13) of them understood as circles around the clock's center and 12(13) understood as circles defined by moving outward in the 12 clock directions. One can now explore the visualization to determine whether it captures relevant facts concerning notes, scales and harmony.

The next figure is based on a "row- and column-cyclic matrix representation" of notes.

c	c	c	c	c	c	c	c	c	c	c	c	c	
0	0	0	0	0	0	0	0	0	0	0	0	0	+12
c	b	A	a	G	g	F	f	e	D	d	C	c	
0	11	10	9	8	7	6	5	4	3	2	1	0	+11
c	A	G	F	e	d	c	A	G	F	e	d	c	
0	10	8	6	4	2	0	10	8	6	4	2	0	+10
c	a	F	D	c	a	F	D	c	a	F	D	c	
0	9	6	3	0	9	6	3	0	9	6	3	0	+9
c	G	e	c	G	e	c	G	e	c	G	e	c	
0	8	4	0	8	4	0	8	4	0	8	4	0	+8
c	g	d	a	e	b	F	C	G	D	A	f	c	
0	7	2	9	4	11	6	1	8	3	10	5	0	+7
c	F	c	F	c	F	c	F	c	F	c	F	c	
0	6	0	6	0	6	0	6	0	6	0	6	0	+6
c	f	A	D	G	C	F	b	e	a	d	g	c	
0	5	10	3	8	1	6	11	4	9	2	7	0	+5
c	e	G	c	e	G	c	e	G	c	e	G	c	
0	4	8	0	4	8	0	4	8	0	4	8	0	+4
c	D	F	a	c	D	F	a	c	D	F	a	c	
0	3	6	9	0	3	6	9	0	3	6	9	0	+3
c	d	e	F	G	A	c	d	e	F	G	A	c	
0	2	4	6	8	10	0	2	4	6	8	10	0	+2
c	C	d	D	e	f	F	g	G	a	A	b	c	
0	1	2	3	4	5	6	7	8	9	10	11	0	+1
c	c	c	c	c	c	c	c	c	c	c	c	c	
0	0	0	0	0	0	0	0	0	0	0	0	0	+0

The cyclic matrix above represents notes by modulo-12 row/column indices via a simple notation, i.e., (0,1,2,3,4,5,6,7,8,9,10,11) = (c, c#, d, d#, e, f, f#, g, g#, a, a#, b) = (c, C, d, D, e, f, F, g, G, a, A, b). Rows +1 and +11 shows chromatic scales and rows +5 and +7 define the circle-of-fifths scale. The underlined groups of notes in the +0, +1, +2, ..., +10, +11, +12 rows are consisting of 1 (+0, +12), 2 (+6), 3 (+4, +8), 4 (+3, +9), 6 (+2, +10) and 12 (+1, +5, +7, +11) notes and are repeated in the respective rows. The number of half-tones, 12, equals 1+1+1+1+1+1+1+1+1+1+1+1, 2+2+2+2+2+2, 3+3+3+3, 4+4+4, 5+7=7+5 and 6+6, determining the numbers of notes in the underlined note groups. This matrix-/torus-based representation exhibits a "border periodicity" of 12 in all rows and all columns, i.e., all 4 periodic borders of the matrix representation have the same root note and index, c and 0. Torus visualization and cyclic matrix visualization are equivalent.

Similarity Based on Overtones. It is important to understand why the "overtone series" (harmonic series) of a note's frequency relative to the base or fundamental frequency f (in Hertz) of a root note defines the degree of consonance or dissonance when playing two notes simultaneously based on 2-note chords, 3-note chords, 4-note chords or other chords. We consider the similarity of the overtone series of two (or more) notes when played as chords.

First, the overtone (frequency) series of a root note having base (fundamental) frequency f is the series 2f, 3f, 4f, ..., i.e., (all) integer multiples of frequency f. "Wavelengths" one can associate with the base and overtone frequencies (harmonics) are the reciprocal quantities 1/f, 1/(2f), 1/(3f), 1/(4f), ... The frequency sequence given by a base frequency of a root note and its octaves (above) is defined by repeatedly doubling the frequency of the base/root note, i.e., 1f, 2f, 4f, ..., 64f, 128f, keeping in mind that a keyboard with 88 keys covers 7 octaves above the lowest A and the lowest C. Second, "just intonation" - one of several methods to establish frequency ratios for note pairs required for tuning an instrument - is often approximated or defined by using simple ratios for note pairs consisting of the base note, e.g., C1, and a second note from the set {C1, D1, Eb1, E1, F1, G1, A1, B1, C2}, for example. Frequency ratios are approximated by small integer ratios: C1:C1 = 1/1, D1:C1 = 9/8, Eb1:C1 = 6/5, E1:C1 = 5/4, F1:C1 = 4/3, G1:C1 = 3/2, A1:C1 = 5/3, B1:C1 = 15/8 and C2:C1=2/1, where one moves upward from C1 (root note) to C2 (octave above). We consider all 12 notes of the chromatic scale. One can generate a table (shown below) of overtone frequencies that are expressed as multiples of the base frequency f. The table shows the factors for the 12-tone scale

defined by half-tone steps. In other words, the numbers are the factors by which f must be multiplied.

Legend for Table below. The 'I' column shows the 13 interval (frequency) ratios relative to a chosen first note (1), e.g., the note C1 having frequency f. Frequency ratios are shown in the 'R' column, where common, only simple frequency ratios are provided for 13 chromatic scale ratios: 1 = unison, m2 = min2nd, M2 = maj2nd, M3 = maj3rd, 4 = 4th, TT = tritone (= dimin5th), m6 = min6th, M6 = maj6th, m7 = min7th, M7 = maj7th and 8 = octave. The 'R' column values are ratios of small integers, merely approximating values used for real tuning. More complicated ratios are used in practice, e.g., for the common 12-tone equal temperament tuning. The table provides 'R' values considered often.

I	R	2nd	3rd	4th	5th	6th	...	Overtone as Integer Multiple of 'R'				
1[C1]	1:1	2[C2]	3[G2]	4[C3]	5[E3]	6[G3]	7	8[C4]	9	10[E4]	11	12[G4]
m2$_{[Db1]}$	17:16	...	...	...	...	...	...	...	...	...	...	...
M2$_{[D1]}$	9:8	...	...	...	...	...	...	9	...	...	...	...
m3[Eb1]	6:5	12/5$_{[Eb2]}$	18/5	24/5$_{[Eb3]}$	6[G3]	36/5	42/5	48/5$_{[Eb4]}$	54/5	12[G4]	66/5	72/5
M3[E1]	5:4	10/4$_{[E2]}$	15/4	5[E3]	25/4	30/4	35/4	10[E4]	45/4	50/4	55/4	15
4[F1]	4:3	8/3$_{[F2]}$	4[C3]	16/3$_{[F3]}$	20/3	8[C4]	28/3	32/3$_{[F4]}$	12[G4]	40/3	44/3	16[C5]
TT$_{[Gb1]}$	36:25	...	...	...	...	...	...	...	...	...	...	...
5[G1]	3:2	3[G2]	9/2	6[G3]	15/2	9	21/2	12[G4]	27/3	15	33/3	18
m6$_{[Ab1]}$	8:5	...	...	...	8[C4]	...	...	...	...	16[C5]	...	...
M6[A1]	5:3	10/3$_{[A2]}$	5[E3]	20/3$_{[A3]}$	25/3	10[E4]	35/3	40/3$_{[A4]}$	15	50/3	55/3	20[E5]
m7$_{[Bb1]}$	9:5	...	...	...	9	...	...	...	...	18	...	...
M7$_{[B1]}$	15:8	...	...	...	...	...	...	15	...	...	...	...
8[C2]	2:1	4[C3]	6[G3]	8[C4]	10[E4]	12[G4]	14	16[C5]	18	20[E5]	22	24[G5]

Numerical 'R' values: 1, 1.063, 1.125, 1.2, 1.25, 1.333, 1.44, 1.5, 1.6, 1.667, 1.8, 1.875, 2

C1 has frequency f. C1's above-octaves have frequencies 2f(C2), 4f(C3), 8f(C4), 16f(C5), ..., indicated in the table after the numbers 2, 4, 8 and 16, respectively. Accordingly, G1(1.5f) and its above-octaves G2(3f), G3(6f), G4(12f) and G5(24f) are also indicated, also defined by frequency doubling, i.e., frequencies 1.5f, 3f, 6f, 12f and 24f, respectively. Other overtones are shown in "[ ]" brackets. The table above defines overtone (OT) similarity:

- The OT frequencies of 1[C1] having base frequency f are 2f, 3f, 4f, ...
- Every OT frequency of 8[C2] is an OT frequency of 1[C1].
- Every 2nd OT frequency of 5[G1] is an OT frequency of 1[C1].
- Every 3rd OT frequency of 4[F1] is an OT frequency of 1[C1].
- Every 3rd OT frequency of M6[A1] is an OT frequency of 1[C1].
- Every 4th OT frequency of M3[E] is an OT frequency of 1[C1].
- Every 5th OT frequency of m3[Eb] is an OT frequency of 1[C1].
- Every 5th/5th/8th/8th/16th/25th OT of m6[Ab1]/m7[Bb1]/M2[D1]/M7[B1]/m2[Db1]/TT[Gb1] is OT of 1[C1].

Considering these facts, one can define a "similarity of notes" – and thus a degree of consonance or dissonance – using the degree of "agreement in OT frequencies" of two notes. Relative to the chosen base note 1[C1], one can order notes in decreasing order of sharing OT frequencies. To define an order for the 7 notes 1[C1], m3[Eb1], M3[E1], 4[F1], 5[G1], M6[A1] and 8[C2], one can calculate their "OT series agreement degrees," given as 1/1=1.0, 1/5=0.2, 1/4=0.25, 1/3=0.33, 1/2=0.5, 1/3=0.33 and 1/1=1.0, respectively. Thus, one obtains the ordered sequence 1[C1], 2[C2], 5[G1], 4[F1], M6[A1], M3[E1] and m3[Eb1] when sorting in decreasing agreement degree. This order reflects the degree of consonance that one can associate with the implied 2-note chords. In this example, the degree of consonance is decreasing for the seven considered 2-note chords according to this order: C1C1, C1C2, C1G1, C1F1, C1A1, C1E1, C1Eb1. More complicated definitions of OT similarity can be used, e.g., to obtain different consonance degrees for chords C1F1 and C1A1.

Furthermore, by using the degree of agreement in OT frequencies, one can calculate a measure of consonance for all possible 2-note chords being note pairs with notes chosen from the set {C1, Db1, D1, Eb1, E1, F1, Gb1, G1, Ab1,

A1, Bb1, B1, C2}. One can ask why this might be desirable from a practical perspective. The answer is given by the fact that the frequency ratio of note pairs of two notes being a specific number of half-tones apart from each other is generally not constant. In other words, the practically used frequency ratio Db1:C1 can be slightly different from the frequency ratio D1:Db1, even though the chords C1Db1 and Db1D1 both consist of note pairs having a half-note distance of one. The reason for the frequency ratios Db1:C1 and Db2:C2 being different is the need to enforce certain frequency ratios to approximate an "overall ideal tuning" for an instrument and room environment.

A row of a table providing all note pair "OT series agreement degrees" would be associated with the root note of 2-note chords, e.g., one could use rows from C1 to C2. A column entry in a row would define the note played above and together with the root note of that row, e.g., the notes C1, Db1, D1, ... and C2 above the note C1. Such a table could be helpful for formally defining and quantifying consonance of 2-note chords. A possible table layout is sketched below for the 12-tone scale with notes C1, Db1, D, ..., C2:

C1	Db1	D1	Eb1	E1	F1	Gb1	G1	Ab1	A1	Bb1	B1	C2
C1C1	C1Db1	C1D1	C1Eb1	C1E1	C1F1	C1Gb1	C1G1	C1Ab1	C1A1	C1Bb1	C1B1	C1C2
1.00	0.06	0.13	0.20	0.25	0.33	0.04	0.50	0.20	0.33	0.20	0.13	1.00
	Db1Db1	Db1D1	Db1Eb1	...	...	...	...	...	...	...	...	Db1C2
		D1D1	D1Eb1	...	...	...	...	...	...	...	...	D1C2
			Eb1Eb1	...	...	...	Eb1G1	Eb1Ab1	Eb1A1	...	...	Eb1C2
				E1E1	...	...	...	E1Ab1	E1A1	...	...	E1C2
					F1F1	...	...	...	F1A1	...	...	F1C2
						Gb1Gb1	...	...	...	...	...	Gb1C2
							G1G1	...	...	...	...	G1C2
								Ab1Ab1	...	...	...	Ab1C2
									A1A1	...	...	A1C2
										Bb1bb1	...	Bb1C2
											B1B1	B1C2
												C2C2

Legend. The F1A1 entry is the value that represents the degree of consonance of the 2-note chord with root note F1 and second note A1, using a measure of similarity for the 2 notes based on their frequency overtone series. (Values for C1C1, C1Eb1, C1E1, C1F1, C1G1, C1A1 and C1C2 are provided.)

Note. Equal temperament frequencies for A4-A5 [Hz] are: 440, 466.2, 493.9, 523.3, 554.4, 587.3, 622.3, 659.3, 698.5, 740.0, 784.0, 830.6, 880. **Frequency spectra** of chord notes show consonance.

Such 2-note chord consonance values can be used to determine the degrees of consonance of 3-/4-/5-note chords. For example, the consonance degree of the chord CEbG (with m3rd, M3rd, and 5th 2-note chords) can be a trivariate function, a combination of consonance values of the 2-note chords CEb, EbG and CG, e.g., the sum 0.2+0.25+0.5=0.95. Defining a chord distance can support the creation of tension/resolution chords and smooth progressions.

It is noteworthy that the circle-of-fifths order of keys (... Bb-F-C-G-D- ...) can also be derived and justified by the fact that the perfect fifth above a root note is "maximally root-like," as far as OT series similarity is concerned. This is not immediately evident, since one commonly uses the number of scale-inherent triads (or triad notes) shared by different keys as the criterion for ordering keys, leading to the circle of fifths. Nevertheless, one can explain the circle-of-fifths order as a "natural order" based on OT series similarity as well, i.e., OT series similarity can be used as a basis for ordering, numbering and indexing notes and keys.

Mathematical Definition of Note/Key Frequencies. Ideally, the frequency of the octave above a root note has twice the frequency of the root note, i.e., given a root note frequency f0 (in Hertz), the sequence of octaves above the root is the frequency series 2f0, 4f0, 8f0, ... Thus, the mathematical function expressing this law is an exponential function F implied by the geometric progression of the frequencies of octaves, represented by this formula:

F(n) = (2^n) f0, n = 0, 1, 2, 3, ...

F(0) = 1 f0, F(1) = 2 f0, F(2) = 4 f0, F(3) = 8 f0, F(4) = 16 f0, F(5) = 32 f0, ...

For example, when defining f0 as f0 = 27.5, the function F(n) generates the ideal frequencies of the eight notes/keys A0, A1, A2, A3, A4, A5, A6 and A7 of an 88-key piano: F(0) = 27.5, F(1) = 55, F(2) = 110, F(3) = 220, F(4) = 440 = A4, F(5) = 880, F(6) = 1760 and F(7) = 3520. To insert the frequencies of 11 half-tones between a root frequency (f0) and the frequency of the octave above it (2f0), for example, one can use the function F to obtain the ideal frequencies of 13 half-tones subdividing the octave accordingly:

F(i/12) = (2^(i/12)) f0 = 12thRoot(2^i) f0, i = 0, 1, 2, …, 12.

F[0] = F(0/12) = 12thRoot(1) f0 = f0, F[1] = F(1/12) = 12thRoot(2) f0 = 1.06 f0,
F[2] = F(2/12) = 12thRoot(4) f0 = 1.12 f0, …, F[12] = F(12/12) = 12thRoot(4096) f0 = 2 f0.

The ratio of frequencies F[i+1] and F[i] of a half-tone pair defined by a half-tone with index i and the next half-tone with index i+1 is the quotient **R** given as

F[i+1] / F[i] = F((i+1)/12) / F(i/12)
= (2^((i+1)/12) f0) / (2^(i/12) f0) = (2^((i+1)/12)) / (2^(i/12))
= 12thRoot(2^(i+1)/2^i) = 12thRoot(2) = 1.059463 …

F[i] = R^i f0, i = 0, …, 12. R^i values are 1, 1.059, 1.122, 1.189, 1.26, 1.335, 1.414, 1.498, 1.587, 1.682, 1.782, 1.888, 2. Ratios of small integers can approximate R^i: R^0 = 1/1 = 1, R^1 = 17/16 = 1.063, R^2 = 9/8 = 1.125, R^3 = 6/5 =1.2, R^4 = 5/4 =1.25, R^5 = 4/3 = 1.333, R^6 = 36/25 = 1.44, R^7 = 3/2 = 1.5, R^8 = 8/5 = 1.6, R^9 = 5/3 = 1.667, R^10 = 9/5 = 1.8, R^11 = 15/8 = 1.875, R^12 = 2/1 = 2. **F(j) = 27.5 * 12thRoot(2^j)**, j = 0, …, 87, defines frequencies F(0), …, F(87) for an 88-key piano in [Hz]. Nevertheless, in practice one usually must deviate from such an ideal set of frequencies for tuning - for a variety of reasons to achieve the most pleasing, most desirable sound of an instrument. If it were possible to express all constraints and sound optimization goals via a mathematical "cost function," one might be able to utilize well-established methods for solving an implied optimization problem, e.g., least-squares or combinatorial optimization methods. Ultimately, instrument characteristics, artistic goals, room/hall, surrounding environment and human ears guide a final tuning.

Church Modes, Pentatonic Scales etc. Using the standard 12-note chromatic scale as a foundation, there are 6 + 5 + 4 + 3 + 2 + 1 = 21 ways to express the number 12 (= number of half-tones in the chromatic scale) as a sum of five '2's (representing five whole-tone steps) and two '1's (representing two half-tone steps). The seven church mode scales are a subset of scales of these 21 possibilities (heptatonic scales). The matrix below shows these 21 possibilities as 7-digit "words," all consisting of five '2's and two '1's. The seven church mode scales are underlined, and their names are included. Example church mode scales are given for every mode, using the notes in the set {C, D, E, F, G, A, B}. The Aeolian and Ionian mode scales are the common natural minor and major scales.

2222211	**2222112**	**2221122**	**2211222**	**2112222**	**1122222**
	2222121	**2221212**	**2212122**	**2121222**	**1212222**
		2221221	**2212212**	**2122122**	**1221222**
		Lydian	**Mixolydian**	**Aeolian**	**Locrian**
		FGABCDEF	**GABCDEFG**	**ABCDEFGA**	**BCDEFGAB**
			2212221	**2122212**	**1222122**
			Ionian	**Dorian**	**Phrygian**
			CDEFGABC	**DEFGABCD**	**EFGABCDE**
				2122221	**1222212**
					1222221

Relative to the standard 12-tone chromatic scale, pentatonic scales use five (instead of seven) notes per octave. Various pentatonic scales exist. For example, five subsequent black keys on a piano keyboard determine pentatonic scales. Further, the notes N1 = C#|Db, N2 = D#|Eb, N3 = F#|Gb, N4 = G#|Ab and N5 = A#|Bb can be played as five different pentatonic scales, i.e., as N1, N2, N3, N4 and N5; as N2, N3, N4, N5 and N1; as N3, N4, N5, N1 and N2; as N4, N5, N1, N2 and N3 or, finally, as N5, N1, N2, N3, and N4. For these five scales one can define "number-of-half-tone-steps words," i.e., the "words" 23223, 32232, 22323, 23232 and 32322, respectively. Generally, 10 possibilities exist to arrange three '2's and two '3's in a 5-digit "word," defining the formally possible pentatonic scales. The sketch below shows all of them.

22233	**22323**	**23223**	**32223**
	22332	**23232**	**32232**
		23322	**32322**
			33222

The five possibilities for the notes N1, N2, N3, N4 and N5 used in the example discussed above are underlined n the left sketch.

The digits that constitute these "words" sum to 12. The steps to be taken sequentially to move upward on one of these pentatonic scales are steps

stepping either two or three half-tones upward. A ‘2’ implies a step consisting of 2 half-tones, and a ‘3’ implies a step consisting of 3 half-tones.

Many more families of scales can be constructed. Classical music, pop, rock, blues and jazz are genres using chromatic, heptatonic, pentatonic and hexatonic (six-tone) scales, including the hexatonic scale defined by six whole-tones.

Relationships between Music Theory, Mathematics, Physics and Other Areas. The presented thoughts are interesting, as they point out relationships between basic and simple principles encountered in Western music theory as well as in mathematical and computational areas, e.g., algebra, arithmetic, geometry and algorithms.

The student who wishes to learn about many foundational topics of algebra, arithmetic, algebra or number theory could consult reference [1], for example, written by Carl Friedrich Gauss (1777-1855) who explored the topics covered in [1] during the time when he lived and studied in Braunschweig (Brunswick) and Göttingen, Germany (mainly between 1795 and 1798).

Hermann von Helmholtz (1821-1894) described a variety of relationships between fundamental aspects of physics (acoustics, sound), music theory, and mathematics as well as anatomy and physiology in [2]. Overtone series, consonance, musical scales and harmony are discussed in detail, from theoretical and experimental viewpoints.

References

[1] Gauss, Johann Carl Friedrich (1801), Disquisitiones Arithmeticae (Arithmetical Investigations), Gerhard Fleischer, Leipzig, Germany.

[2] Helmholtz, Hermann Ludwig Ferdinand von (1863), Die Lehre von den Tonempfindungen als physiologische Grundlage fuer die Theorie der Musik (On the Sensations of Tone as a Physiological Basis for the Theory of Music), Friedrich Vieweg und Sohn, Braunschweig, Germany.

Addendum: Overtone (OT) Frequency Series

The formula **F(i) = 27.5 * 2^(i/12**) = 27.5 * 12thRoot(2^i), i = 0, ..., 87, defines the frequencies F(0), ..., F(87) for all half-tones on an 88-key piano in [Hz], using equal temperament tuning, i.e., the tuning that is commonly used today. The frequency ratio R = F(i+1) / F(i) is a constant. The frequency ratio for all pairs of half-tones next to each other (neighboring half-tones with indices i+1 and i) has the value R = 12thRoot(2) = 1.059463 ...

One can create a table for the representation and comparison of OT frequency series of all half-tones, all "semi-tones." A table with 13 rows - having one row for each of the 13 half-tones in the octave above a fixed root note, counting the root note itself - can provide OT frequency series data for each of the 13 half-tones in the octave. Each of the 13 half-tones serves as the second, upper note of a 2-note chord above the root note. In a specific row, a table can list OT frequency data for the half-tone associated with that row, i.e., data (multiplication factors) that define how OT frequencies of the row-associated half-tone are written as multiples of the fundamental frequency of the root.

When thinking in terms of intervals, such a table provides OT series data for the 13 intervals 1(1st), m2(m2nd), M2(M2nd), m3(m3rd), M3(M3rd), 4(4th), dim5 (TT) (dim5th(tritone)), 5(5th), m6(m6th), M6(M6th), m7(m7th), M(M7th), and 8(8th), which is equivalent to thinking about the 13 2-note chords in an octave. (The abbreviations 'm', 'M' and "dim" stand for minor, major and diminished.) OT frequency series data are represented by the factors required to multiply the root's fundamental frequency f to obtain the OT frequencies of a 2-note chord's upper half-tone. The OT frequencies of a note/tone in a specific row, represented via these factors, are the frequencies being integer multiples of the fundamental frequency of the note/tone associated with that row.

Using a root note with fundamental frequency f, the frequencies of the 13 half-tones above this root determining all possible 2-note chords (intervals 1, m2, M2, m3, M3, 4, dim5(TT), 5, m6, M6, m7, M7 and 8) are f * R^i, i = 0, ..., 12, where R^0, R^1, R^2, ... and R^12 have approximate values 1, 1.059, 1.122, 1.189, 1.26, 1.335, 1.414, 1.498, 1.587, 1.682, 1.782, 1.888 and 2. These

approximate R^i values are shown in the table below in the 1st column. The "OT SPECTRA" table layout without including specific values is sketched first.

F [i,j]	j=1	j=2	**j**	j=3	...	**OT SPECTRA**
i=0	1 R^0 =1	2 R^0 = 2	j R^0 = j	3 R^0 = 3	...	The table rows
i=1	1 R^1	2 R^1	j R^1	3 R^1	...	**F [i1,j]**, **F [i2,j]**,
i	1 R^i	2 R^i	**j * R^i = F [i,j]**	3 R^i	...	j = 1, 2, 3, ...
i=12	1 R^12 = 2	2 R^12 = 4	j R^12 = 2j	3 R^12 = 6	...	are **2 spectra**.

To calculate OT similarity, one must properly define the distance of 2 spectra.

The following table provides detailed data for OT frequencies.

Chord	1st	2nd	3rd	4th	... Overtones of 2nd Note of Chord as Multiple of f							
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
m2	1.059	2.119	3.178	4.238	5.297	6.357	7.416	8.476	9.535	10.594	11.654	12.714
M2	1.122	2.245	3.367	4.490	5.612	6.735	7.857	8.980	10.102	11.225	12.347	13.470
m3	1.189	2.378	3.568	4.757	5.946	7.135	8.324	9.514	10.703	11.892	13.081	14.270
M3	1.260	2.520	3.780	5.040	6.300	7.560	8.819	10.079	11.339	12.599	13.859	15.119
4	1.335	2.670	4.005	5.339	6.674	8.009	9.344	10.679	12.014	13.348	14.683	16.018
dim5(TT)	1.414	2.828	4.243	5.657	7.071	8.485	9.899	11.314	12.728	14.142	15.556	16.971
5	1.498	2.997	4.495	5.993	7.492	8.990	10.488	11.986	13.485	14.983	16.481	17.980
m6	1.587	3.175	4.762	6.350	7.937	9.524	11.112	12.699	14.287	15.874	17.461	19.049
M6	1.682	3.364	5.045	6.727	8.409	10.091	11.773	13.454	15.136	16.818	18.500	20.182
m7	1.782	3.564	5.345	7.127	8.909	10.691	12.473	14.254	16.036	17.818	19.600	21.382
M7	1.888	3.775	5.663	7.551	9.439	11.326	13.214	15.102	16.990	18.877	20.765	22.653
8	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**

The values in every i^{th} row of the table are integer multiples of the value in the 1^{st} column of the same row. The i^{th} row shows the values in the 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, ... columns (j = 1, 2, 3, 4, ...) being the factors one must use to multiply the root note's fundamental frequency f by, to obtain the frequency of the j^{th} OT of the note/tone associated with the i^{th} row. Thus, OT frequencies are expressed by factors serving as multipliers to be applied to the root note's fundamental frequency f. Underlined values in the table highlight values being integers or values being very close to integers, which are of interest.

The integers in the table being powers-of-2 (1, 2, 4, 8, 16, 32, 64, 128, ...) represent octaves of a root note with fundamental frequency f. If f were the fundamental frequency of the root note C1, then C1's octaves C2, C3, C4, C5, C6, C7, C8, ... would have fundamental frequency values 2f, 4f, 8f, 16f, 32f, 64f, 128f, ..., respectively. Therefore, one can use the table to determine whether C1's octaves are part of the OT frequency series represented in rows m2, M2, m3, ..., M7 or 8. The powers-of-2 values 4, 8 and 16 occur ("nearly exactly") in row 4, the power-of-2 values 8 and 16 also occur in row m6, and the power-of-2 value 16 also occurs in row m7. This means that C3, C4 and C5 arise in the OT frequency series of C1's 4^{th}, that C4 and C5 arise in the OT frequency series of C1's $m6^{th}$, and that C5 arises in the OT frequency series of C1's $m7^{th}$. Thus, when associating the note F1 with row 4, OTs 3, 6 and 12 of F1 are C1's octaves C3, C4 and C5, respectively. Analyzing the table in this way provides another means for determining similarity of OT frequency series.

The following example explains the table's meaning in detail.

Example. We choose the note A with fundamental frequency f = 440[Hz] as root note and consider the perfect 5^{th} above A, i.e., the note E with frequency R^7 * 440[Hz] = 1.498 * 440[Hz] = 659.12[Hz]. The values listed in row '5' (E) are the factors one must multiply the root note's fundamental frequency f by, to obtain the frequencies of the OT frequency series of E. The row means: E's OT frequencies (E's fundamental frequency viewed as 1^{st} OT frequency of E) are obtained by multiplying f with the factors 1.498, 2.997, 4.495, 5.993, 7.492, 8.990, 10.488, 11.986, 13.485, 14.983, 16.481, 17.980, ... Here, the

underlined values are close to the integers 3, 6, 9, 12, 15, 18, … In other words, every second OT frequency of E is an OT frequency of A. Thus, the frequencies that A and E share as common OT frequencies are 3f, 6f, 9f, 12f, 15f, 18f, … The two notes' OT frequency series are "very similar," and one can consider the 2-note chord AE as "very consonant."

This example effectively compares the notes A and E via their OT frequency series. One can understand such a comparison as a comparison of two spectra. Generally, a spectral analysis and comparison of notes/sounds first represents notes/sounds to be compared by their frequency spectra and subsequently compares the spectra to determine the degree of similarity of the notes/sounds. Notes/sounds can be viewed as being similar when their frequency spectra are similar, i.e., when the number of equal OT frequencies - number of "nearly equal OT frequencies" - is high and the amplitudes are of similar magnitude. It is not trivial to define a measure of "distance of spectra," which is a crucial foundation for a viable method for chord comparison.

A "well-refined" spectral analysis and comparison of OT frequency series data makes possible a quantitative assessment of the degree of similarity and thus of consonance of chords. Consequently, one could rank and order chords by using a measure for spectral similarity as an ordering criterion. Considering the example described above, one has 13 OT spectra for A, where the 13 tones in the set {A(440Hz), Bb, B, C, Db, D, Eb, E, F, Gb, G, Ab, A(880Hz)} serve as upper notes of 13 2-note chords with root A(440Hz). One must establish a viable means for quantifying OT spectral distance numerically to properly reflect the "distance and similarity as perceived by the human ear." The goal would be to establish a mathematically sound, experimentally verified and therefore practically usable technique for ranking 2-/3-/4-note chords, and other chords, to guide, for example, the creation of smooth progressions.

Tuning methods different from equal-temperament tuning do not enforce a constant half-tone frequency ratio of $R = 12^{th}\text{Root}(2)$ for all half-tone neighbors having indices i+1 and i on a piano keyboard. To simplify the above discussion concerning OT frequency series data - and to emphasize the fact that modern

instrument tuning uses primarily equal temperament - the discussion focuses on a constant ratio R. Non-standard tuning methods, e.g., well-temperament tuning, use half-tone ratios that can deviate more or less from the constant ratio R, creating different values for the frequency ratio of neighboring half-tone pairs (indices i+1 and i defining neighbors) in one octave, or for the frequency ratio of a chord's two notes when played in different octaves. Tunings not based on equal temperament are often viewed as tunings that produce different "colors, emotions or sensations," which one can associate with the 12 major keys, for example. Consequently, when transposing a melody from one key to another key, the melody heard in the two transposed versions on a piano not tuned with equal temperament tuning could be perceived as having different "colors," leading to slightly different listening experiences.

As a related anecdote, the pianist Arturo Benedetti Michelangeli was so sensitive to and critical of tuning and sound. He often did not perform when his high expectations regarding instrument tuning and performance hall sound environment were not "perfectly" met.