A FEW SIMPLE REFLECTIONS ON HARMONY:

Keys and Scales,
Melody and Accompaniment,
Stable and Unstable/Tension Chords,
Resolutions and Progressions,
Transitions and Modulations
with an
Appendix on Geometrical Transformations

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0. TOPICS

The following sentences describe the related topics partially addressed here:

A melody implies a key or a given key defines the notes for a melody.
A note or chords are desired and used to accompany the melody.
An accompaniment can be based, for example, on triads.
A viable method for choosing these triads considers the three major, the three minor and the one diminished triad of the used scale.
A major-to-minor-triad or minor-to-major-triad shift leads to a “mood change.”
A chord can be viewed or perceived as a stable or unstable/tension chord.
Example: C-major has stable triads I=C, ii=Dm, iii=Em, IV=F, V=G, vi=A.
An unstable/tension chord requires/calls for a resolution to a stable chord.
Example: C-major tension chords are viidim=B-D-F, s2=C-D-G, s4=C-F-G.
A progression of chords can transition/move in several different ways.
Example: Stable-to-tension (-to-stable) or tension-to-stable (-to-tension).
A modulation is used to move a current key (tonic center) to a new one.
A modulation can abruptly move to a new key or perform a “smooth shift.”
A modulation can involve many complex chords to move to a new key.

Only some of the issues alluded to are sketched at a high level in the notes.
General principles are discussed; simple examples are included as well.
Readers will likely identify errors. Hopefully, they will kindly forgive them.

Ohne Musik wäre das Leben ein Irrtum.
(Without music, life would be a mistake.)
Friedrich Wilhelm Nietzsche

1. DEFINING 64 CHORDS WITH ROOT C

The 64 chords using notes from {C, D, Eb, E, F, Gb, G, G#, A, Bbb, Bb, B} with each note/chord playable in several octaves and in all inversions:

C D Eb,E F Gb,G,G# A Bbb,Bb,B Row-defined chords

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———————— 6C0 = 1 ————————

C **C**

———————— 6C1 = 6 ————————

C D

C EbE **Cmin, Cmaj**

C F

C GbGG#

C A

C BbbBbB

———————— 6C2 = 15 ————————

C D EbE

C D F

C D GbGG#

C D A

C D BbbBbB

C EbE F

C EbE GbGG# **Cdim, Cmin, Cmaj, Caug**

C EbE A

C EbE BbbBbB

C F GbGG#

C F A

C F BbbBbB

C GbGG# A **Aug Italian 6th Ab-C-F#|Gb**

C GbGG# BbbBbB

C A BbbBbB

———————— 6C3 = 20 ————————

C D EbE F

C D EbE GbGG#

C D EbE A

C D EbE BbbBbB

C D F GbGG#

C D F A

C D F BbbBbB

C D GbGG# A **Aug French 6th Ab-C-D-F#|Gb**

C D GbGG# BbbBbB

C D A BbbBbB

C EbE F GbGG#

C EbE F A

C EbE F BbbBbB

C		**EbE**		**GbGG#**	**A**		**C6 = Am7[1]** ([1] = 1st inversion)
							Aug German 6th Ab-C-Eb-F#IGb
C		**EbE**		**GbGG#**		**BbbBbB**	**Cfulldim7, Chalfdim7, …**
							Cmmaj7, Cm7, C7, Cmaj7
C		EbE			A	BbbBbB	
C			F	GbGG#	A		
C			F	GbGG#		BbbBbB	
C			F		A	BbbBbB	
C				GbGG#	A	BbbBbB	
— — — — — — — — — —			**6C4 = 1**			— — — — — — — — —	
C	D	EbE	F	GbGG#			
C	D	EbE	F		A		
C	D	EbE	F			BbbBbB	
C	D	EbE		GbGG#	A		
C	**D**	**EbE**		**GbGG#**		**BbbBbB**	**Cmaj9, C9, …**
C	D	EbE			A	BbbBbB	
C	D		F	GbGG			
C	D		F	GbGG#		BbbBbB	
C	D		F		A	BbbBbB	
C	D			GbGG#	A	BbbBbB	
C		EbE	F	GbGG#	A		
C		EbE	F	GbGG#		BbbBbB	
C		EbE	F		A	BbbBbB	
C		EbE		GbGG#	A	BbbBbB	
C			F	GbGG#	A	BbbBbB	
— — — — — — — — — —			**6C5 = 6**			— — — — — — — — —	
C	D	EbE	F	GbGG#	A		
C	**D**	**EbE**	**F**	**GbGG#**		**BbbBbB**	**Cmaj11, C11, …**
C	D	EbE	F		A	BbbBbB	
C	D	EbE		GbGG#	A	BbbBbB	
C	D		F	GbGG#	A	BbbBbB	
C		EbE	F	GbGG#	A	BbbBbB	
— — — — — — — — — —			**6C6 = 1**			— — — — — — — — —	
C	**D**	**EbE**	**F**	**GbGG#**	**A**	**BbbBbB**	**Cmaj13, C13, …**

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Legend. The "n-Choose-k function" (nCk) is defined as nCk = n! / (k! x (n-k)!).
Thus, 6C0+6C1+6C2+6C3+6C4+6C5+6C6=1+6+15+20+15+6+1=64.

By lowering and/or raising notes via 'b', "bb" or '#' — considering the notes E, G and/or B, for example — one obtains minor, diminished or augmented chords, needed for modulation to another key and/or the creation of tension chords. Specifically, the table above uses the "generalizations" of the the notes E, G and B to "note sets," i.e., these notes define one set of two notes (in the case of E) and two sets of three notes (in the cases of G and B):

E —> { Eb, E } G —> { Gb, G, G# } B —> { Bbb, Bb, B }

Note. The book *"Theory of Harmony"* [4] by Arnold Schoenberg, published first in 1911, explores harmony in detail. Jean-Philippe Rameau [1], Hugo Riemann [2], Heinrich Schenker [3] and Gottfried Weber [5] also published books on harmony.

2. TRANSITIONS AND RESOLUTIONS FOR CHORDS WITH ROOT C

Simple transition (stable-to-stable) and resolution (tension-to-stable) chords:

C-chord	Resolution chords	Transition chords
C		**C, F, G, A**
C-D	**C-E**	
C-Eb		**C-E, C-Eb**
C-E		**C-E, C-Eb**
C-F		**C-E, C-F, C-G, C-A**
C-Gb	**C-G**	
C-G		**C-E, C-G**
C-G#	**C-F, C-G, C-A**	
C-A		**C-F, C-A**
C-Bb	**C-F, C-A**	
C-B	**C**	
C-D-G	**C, Cm** (triads)	
C-Eb-Gb	**C, Cm**	
C-Eb-G		**C, Cm, Fm, Ab, Bb**
C-E-G		**C, Dm, G**
C-E-G#	**F, Am**	
C-F-G	**C, Cm**	
C-Eb-Gb-Bbb	**Db(m), E(m), G(m), Bb(m)**	
C-Eb-Gb-Bb	**Db(m), G7b9** (G-B-D-F-Ab)	
C-Eb-G-Bb	**F7, Bbmaj7**	
C-Eb-GB	**Cm, Cm6, Cm7**	
C-Eb-F#-Ab[3] [Aug German 6th]	**C, Eb, G**	
C-E-G-A		**C, A**
C-E-G-Bb	**F, Fm** (deceptive: Db, Am)	
C-E-G-B	**Dm7, Fmaj7, Fm, Am7, G, G7**	(C, Cmaj7)
C-F-G-Bb	**C, C7** (resolving to F)	
C-F#-Ab[2] [Aug Italian 6th]	**C, Cm, Db, Dbm, G**	
C-D-E-G-Bb	**C, F, Fmaj7, Fm7**	
C-D-E-G-B	**C, Cmaj7, E**	(Cmaj9)
C-D-F#-Ab[3] [Aug French 6th]	**C, G**	

C-D-E-F-G-Bb	**C, C7, F**
C-D-E-F-G-B	**C, C6, Fmaj7, Fmaj9**
C-D-E-F-G-A-Bb	**C, Fmaj9, F6/9, Fm9**
C-D-E-F-G-A-B	**C, Cmaj7, Em7, Am7**

3. PROGRESSIONS FOR MAJOR AND MINOR KEYS

Progressions in major or minor keys (mainly) consist of chord sequences chosen from a key's associated major, minor and diminished chords, for example. Therefore, a simple progression in a major or minor key could consider the seven triads I, ii, iii, … (major key) or I, ii0 (iidim), III, … (minor key) to define a progression consisting only of key-defined triads. In the case of a minor key, one must keep in mind that the three minor scales (natural, harmonic and melodic) and leading-tone requirements have to be considered. Naively, one could simply determine theoretically possible progressions of only two chords, i.e, being of the kind L—> R, where a left chord L progresses/moves to a right chord R. Further, one could also assume that a larger progression consisting of more than two chords might merely be a "concatenation" of elementary two-chord L—> R progressions.

Of course, this in not a viable method for constructing "allowable or desirable" progressions. Instead, one must know underlying harmonic relationships between chords in detail, must listen whether an intended effect of a constructed progression is achieved, whether voice-leading principles are satisfied when moving from chord to chord etc. Thus, the tables provided below — showing how two-chord L—> R progressions arise and are found in larger progressions — cannot be viewed as tables that contain the combinable "building blocks" for larger progressions.

The tables merely summarize that these two-chord L—> R progressions appear in commonly used progressions found in compositions. A viable method for establishing good, complex progressions should keep in mind the many harmonic principles that one should or could employ, i.e., the principles and "rules" discussed in the literature on harmony and its use in (Western) classical music and jazz, for example. Nevertheless, one must also remind oneself that "rules" will have to be broken if one wants to be forward-looking, innovative to discover new ways for composition. Other relevant aspects include color, texture, mood (anger, happiness, sadness, …) tonality/atonality, polyphony etc. that one has to consider when constructing progressions. Ultimately, one must play the instrument to determine what a "good" progression is for achieving one's objectives.

Chord progressions commonly associated with major and minor keys:

\ Moving to									
\	I	ii	iii(III)	IV	V(7)	vi	vii0	A6	N6
From \	C	Dm	Em(E)	F	G(7)	Am	Bdim	Ab-C...	Ab-Db-F
I	I	ii	iii	IV	V(7)	vi	vii0	A6	N6
C	C	Dm	Em	F	G(7)	Am	Bdim	Ab-C...	Ab-Db-F
ii	I		iii	IV	V(7)	vi	vii0	A6	N6
Dm	C		Em	F	G(7)	Am	Bdim	Ab-C...	Ab-Db-F
iii	I	ii	iii	IV	V	vi			
Em	C	Dm	Em	F	G	Am			
IV	I	ii	iii		V	vi			
F	C	Dm	Em		G	Am			
V	I	ii		IV	V	vi			
G	C	Dm		F	G	Am			
vi	I	ii	iii(III)	IV	V	vi			
Am	C	Dm	Em(E)	F	G	Am			
vii0	I		iii				vii0		
Bdim	C		Em				Bdim		
A6	I				V(7)			A6	
Ab-C…	C				G(7)			Ab-C...	
N6	I	ii		IV	V(7)				N6
Ab-Db-F	C	Dm		F	G(7)				Ab-Db-F

\ Moving to									
\	i(I)	ii0	III(iii)	iv(IV)	v(V)	VI	VII(vii0)	A6	N6
From \	Am(A)	Bdim	C	Dm(D)	Em(E)	F	G(G#dim)	F-A ...	F-Bb-Eb
======== \									
i	i	ii0	III	iv	v	VI	(vii0)	A6	N6
Am	Am	Bdim	C	Dm	Em	F	(G#dim)	F-A...	F-Bb-Eb
ii0		ii0			(V)	VI			
Bdim		Bdim			(E)	F			
III	i		III	iv(IV)	v(V)	VI	VII		
C	Am		C	Dm(D)	Em(E)	F	G		
iv	i	ii0		iv(IV)	v(V7)	VI	VII		
Dm	Am	Bdim		Dm(D)	Em(E7)	F	G		
v	i	ii0		iv	v(V7)	VI	VII		
Em	Am	Bdim		Dm	Em(E7)	F	G		
VI	i		III	iv	(V7)	VI	VII		
F	Am		C	Dm	(E7)	F	G		
VII	i		III	iv	(V7)	VI	VII		
G	Am		C	Dm	(E7)	F	G		
vii0	i(I)		III(iii)				(vii0)		
G#dim	Am(A)		C(Cm)				(G#dim)		
A6	(I)				(V7)			A6	
F-A...	(A)				(E7)			F-A...	
N6	i(I)			iv	v				N6
Bb[1]=D-F-Bb	Am(A)			Dm	Em				F-Bb-Eb

Legend.	Table rows	Each row of the table lists (some) chords the chord listed in the respective “From” column can move to. It would suffice to simply show a mark ‘*’ in the specific chord columns to indicate that a “From” chord can move to a “Move to” chord; instead, the table provides full names of “Move-to” chords, for clarity.
	I, ii, iii, IV, V, vi, vii0 (Chord), (7)	The 7 degrees of a major scale (vii0=viidim) The chord “(Chord)” does not belong to the considered major or minor key, but “(Chord)” is practically used. The number “(7)” means that a dominant 7th chord can also be used.
	Chord[1] ([2], [3])	1st (2nd, 3rd) inversion of the chord “Chord”
	A6 (notes not in tonic)	Augmented 6th (Ab in C-major, F in A-minor): C-maj: Ab-C-F#|Gb (Italian), Ab-C-D-F#|Gb (French) Ab-C-Eb-F#|Gb (German); A-min: F-A-Eb

		(Italian), F-A-B-Eb (French), F-A-C-Eb (German)
	N6 (notes not in tonic)	Neapolitan 6th (Ab in C-major, F in A-minor): in C-major: Ab-Db-F; in A-minor: F-Bb-Eb
	I, ii0, III, iv, v, VI, VII (vii0)	The 7 degrees of the natural minor scale (vii0=viidim used in harmonic minor scale)

Note. These "from-to" tables can be represented by graphs with directed edges, with an edge's direction reflecting the "from-to" chord relationship. A graph representation involving a large number of progressions generally contains intersecting edges, and comprehension can become hard. Nevertheless, the literature provides a variety of techniques that one can use to ensure that such graph-based representations remain comprehensible.

Next, an extended list of important progressions is provided. The list includes several progressions that contain subsets of two-chord building-block progressions L—> R, where L = R. For example, the Blues progressions include I—> I and also IV—> IV. Of course, the same chord can be repeated multiple times in a progression, unless a melody requires one to change chords. Important aspects not discussed here include time signature, "rhythm," periodicity etc., all influencing the creation of a progression.

Common chord progressions, mainly staying in a major/minor key, ordered according to: I < i < II < ii < ii0 < III < iii < IV < iv < V < v < VI < vi < VII < vii< vii0:

Pop music
I-IV-V
I–V–vi–IV
I-vi-ii-V
I–vi–IV–V (or **I** as last chord)
vi-IV-I-V

Jazz
I–vi–ii–V
I–I7–iv–I–V–I
ii–V–I
ii-v-i
iii-vi-ii-V

Blues
I-I-I-I-I-I-I-I-IV-IV-I-I-V-IV-I-I
I–I–I–I–IV–IV–I–I–V–IV–I–I (or **V** as last chord)
I–V–IV–IV–I–V–I–V

Modal progressions

I-IV-I-V-I-IV-I-V-I (passmezzo moderno progression)
i-V-i-bVII-bIII-bVII-i-V-i-V-i-bVII-bIII-bVII-i-V-i (**Folia** progression)
i–VII–i–V–III–VII–i–V–i (passamezzo antico progression)
i–VII–VI–V or i–iv–III–VII
iv–III–bII–I (**Andalusian** progression/cadence, in flamenco minor I–bVII–bVI–V)

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Other progressions described on Wikipedia, for example:

I-IV-ii-V
I-IV-bVII-IV
I-IV-vii0-iii-vi-ii-V-I (**Circle-of-Fifths** progression, major key)
I-iv-ii-V/I-I7-iv-I-V-I/III7-VI7-II7-V7
I-V-ii-vi-iii-vii0-IV-I (**Circle-of-Fifths** progression, major key)
I-V-IV-IV-I-V-I-V
I–V–vi–iii–IV–I–IV–V (**Pachelbel** progression in C-major: C-G-Am-Em-F-C-F-G)
I-V/bVI-bVI-V/III-III-V-I
I-V-bVII-IV
I-bVII-bVI-bVii
I-vii0-III7-vi-II7-v-I7-IV7-iv-bVII7-iii-VI7-biii-bVI7-ii-V7-I-VI7-ii-V
i-ii0-III-iv-v-VI-VII-i (**Circle-of-Fifths** progression, natural minor key)
i-iv-VII-III-VI-ii0-V-i (**Circle-of-Fifths** progression, minor key)
ii-bIII+-I
ii-V-i
ii-bVII7-I
III7-VI7-II7-V7
III-VII-i-V-III-VII-i-V-i
IV7–#ivdim7–I7/5–I7–IV7–#ivdim7–I7/5–I7–IV7–#ivdim7–I7/5–V7/V/V–V7/V–V7–I7
IV-V-I-vi
V7-III7
V-IV-I
bVI-bVII-I
vi-ii-V-I
bVII-V7-i
vii07/V-V-I

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Legend. For simplicity, instead of using "—>" to denote the progression of a chord to a subsequent chord, the single symbol '-' is used in these modulations.

Note. As done before, i.e., for simple two-chord L—> R progressions, it is possible to construct a "from-to" table for "more-than-two-chords" progressions as well. A representation and visualization would have to show the various possibilities for a specific initial chord Cinit to move to a final chord Cfin via a lengthy progression Cinit—> C2—> C3—> … —> Cfin, involving any number of chords to achieve the.

desired “smooth” transitions. Such a table might show that any chord could “In principle” move to “nearly every” chord and thus become rather large. Therefore, a table-based representation of theoretically possible or practically viable progressions is not necessarily an effective or easily understandable visualization. Instead, one can utilize the concept of *parallel coordinates*, known in i*nformation visualization*, for example, representing and visualizing progressions via polylines. When using this concept, progressions are mapped to “left-to-right polylines,” with chords becoming nodes numbered 1, 2, 3, …, chosen from a used ordered set of chords. By listing chords vertically and aligning the same chords horizontally, one can visualize a set of progressions by a set of (“left-to-right-directed”) polylines connecting the chords 1, 2, 3, … of each progression. A progression’s number of included chords defines its associated polyline’s length. We provide simple examples below.

Major key chords

1		2		3		4		5		6	
I	->	I		I		I		I		I	
ii	->	ii		ii		ii		ii		ii	
iii	->	iii		iii		iii		iii		iii	
IV	->>>->	IV	—>	IV	—>	IV	—>	IV	—>	IV	…
V	->	V		V		V		V		V	
vi	->	vi		vi		vi		vi		vi	
vii0	->	vii0		vii0		vii0		vii0		vii0	

Minor key chords

1		2		3		4		5		6	
i		i		i		I		I		i	
ii0		ii0		ii0		ii0		ii0		ii0	
III		III		III		III		III		III	
iv	—>	iv	—>	iv	—>	iv	—>	iv	—>	iv	…
v		v		v		v		v		v	
VI		VI		VI		VI		VI		VI	
VII		VII		VII		VII		VII		VII	
vii0		vii0		vii0		vii0		vii0		vii0	

Note. This table includes only the 7 (8) “typical” major, minor and diminished chords of the major and natural/harmonic minor scales. In practice, many more chords are used for the construction of progressions. *The lexicographically ordered progressions listed above do not imply that their corresponding polylines do not intersect. The lexicographically ordered list of progressions leads to an order for the polylines, with the polyline of the first progression being the first and the polyline*

of the last progression being the last. One can employ an information visualization technique where the first, second, third, … chords of each progression define their corresponding parallel coordinate points on the vertical chord columns 1, 2, 3, … The resulting visualization is likely to contain "too many" progression-associated polylines, leading to a cluttered visualization. Therefore, one must use methods for "de-cluttering" or visualize fewer progressions and thus fewer polylines.

4. MODULATIONS FROM C MAJOR AND A-MINOR TO OTHER KEYS

Modulations make possible the "smooth," "not-so-smooth" or "abrupt" change from one tonic center to another tonic center. The two centers can be close to each other or rather distant — when using the order of keys as defined by the circle-of-fifths, for example, to determine the distance of two harmonic centers. Transitioning from one tonic center to another requires one to define a "proper," a "good" or a "desirable" chord progression that, in principle, can include chords belonging to any key.

Common Pivot (*Drehpunkt*) **Chord Modulation.** Uses a **chord that exists in starting and destination keys** for a smooth transition. Moves from C major to F major, for example, by using a D minor or A minor chord — which belong to both keys — before moving to the new tonic.
Example: **G**—> D—> **Gm (pivot chord)** —> Cm—> F7—> **Bb**

Common-tone Modulation. Highlights a **tone that acts as a bridge or pivot** between two unrelated, distant chords. Creates a moment where the harmony shifts underneath a static note.
Example: **G**—> **D (sustained tone)**—> G—> Bb—> F—> **Bb** or
G—> **D**—> G—> Bb/D (Gm7/D)—> F/A—> **Bb**

Enharmonic Modulation. Re-interprets spelling of a chord, e.g., a fully diminished 7th or an augmented 6th chord serving as pivot chord, **resolving to** a **distant key**. Allows one to pivot between distant keys that share few notes.
Example: **G**—> C—> D7—> **Eb-Gb-A-C (E-G-Bb-Db) (fulldim7th)**—> …
… —> Ebm7—> F7—> **Bb** (—> Eb—> F—> Bb)

Chromatic Modulation. Moves to new key by **raising (lowering) one or more notes by half steps** (chromatically) via smooth voice leading. Feels unexpected, keeping inner voices closely connected.
Example: **G**—> Em (C)—> **Cm (chromatic pivot chord)**—> F7—> **Bb**

Sequential Modulation. Transposes a **melody or harmonic pattern up or down by** a specific **interval**, e.g., by a major 2nd or minor 3rd, and repeats it. Creates momentum, pulling the ear to the new tonic.
Example: **G**—> Em—> Am—> D7—> **G**—> D—> Em—> C—> …
… —> Cm7 (C#fd7)—> F7—> **Bb**

Direct (Abrupt) Modulation. Moves **straight to new key** without preparation or transition chords. Creates "dramatic" shift — used, for example, in pop music when the final chorus moves up a half step ("gear change'').
Example: **G**—> Gmaj7—> Am7—> D7—> **G**—> **Bb** (—> Cm7—> F7—> Bb)

Note. This list of six common modulation methods is based on Google descriptions.

5. MODULATIONS FROM C-MAJOR AND A-MINOR TO OTHER KEYS

The modulations listed below are "ordered lexicographically." Using the order implied by the circle-of-fifths, for example, would be more appropriate when sorting chords and thus modulations being chord sequences (if at all possible). For C-major and A-minor, the orders used here are, on a high level, the orders given by the relations C < D < E < F < G < A < B and A < B < C < D < E < F < G, respectively, taking proper care of the specific notes of the particular chords.

Possibilities to modulate from C-major / A-minor to major / minor keys, using, for simplicity, 'b' notation even when '#' is formally required:

From C to	Sequence of viable chord progressions for modulation
C	**C—> C**
Cm	**C—> Cm //**
	C—> F—> Fm—> G7—> Cm //
	C—> (F—> G—> C—>) Cm (—> Fm—> G7—> Cm) //
	C—> F—> G7—> Cm
Db	**C—> C#/Db //**
	C—> F#7—> Ab7 //
	C—> Ab (Ab7)—> C#lDb / C—> C-Eb-F-Ab—> C#lDb
Dbm	**C—> Caug—> C# //**
	C—> E (E7)—> C#lDd //
	C—> (F—> G—> C—>) Ab7 —> C#lDb //
	C—> Ab (Ab7)—> C#/Db
D	**C—> Em—> A7s4—> A7—> —> D //**
	C—> F—> A7—> D //
	C—> G—> D //
	C—> Bb—> C—> D
Dm	**C—> F—> A7—> D //**
	C—> G—> A7—> Dm //
	C—> Gm—> A7—> Dm
Eb	**C—> Cm—> Fm—> Bb—> Eb //**
	C—> Ab—> Bb7—> Eb //
	C—> Ab—> Fm—> bb7—> Eb //
	C—> Ab—> Bb7—> Eb
Ebm	**C—> Cmaj7—> Gb (Bb7)—> Ebm //**
	C—> Cm—> Fm—> Bb7—> Ebm //

	C—> F—> G7—> Bdim—> Bb7—> Ebm—> Abm—> Bb7—> Ebm //
	C—> Cm—> Fm—> Bb7—> Ebm
E	C—> Em—> B7—> E //
	C—> F—> B7— E //
	C—> Am—> A (A7)—> B7—> A //
	C—> Am—> B7—> E
Em	C—> F (Fm)—>B7—> Em //
	C—> G—> Am—> B7—> Em //
	C—> Am—> B7—> Em
F	C—> C7—> F //
	C—> Dm—> C7—> F //
	C—> F //
	C—> Gm7—> C7—> F
Fm	C—> C7—> Fm—> C7—> Fm //
	C—> F—> Gm—> C7—> Fm //
	C—> F—> Ab—> C7 (Cfulldim7) —> Fm
F#lGb	C—> G7—> C#7lDb7—> F# //
	C—> G/B—> Db —> C#7lDb7—> F# //
	C—> Am—> Abmaj7—> Bb7—> C#7lDb7—> F#
F#mlGbm	C—> C#/BlDb/B—> F#m/A (Gbm/A)—> F#m (Gym) //
	C—> G/B—> C#7 (Db7)—> F#m/Gbm
G	C—> Cfulldim7—> G //
	C—> F—> G //
	C—> F—> Am—> D (D7)—> G //
	C—> G
Gm	C—> Cm—> Eb—> D7—> Gm //
	C—> F—> Bb—> Eb—> D7—> Gm //
	C—> Am—> Dm—> D7—> Gm
Ab	C—> Cm—> Fm—> Eb7—> Ab //
	C—> E (E7)—> Ebmaj7—> Abmaj7—> Ab
Abm	C—> Caug (E)—> Abmaj7—> Bbm7 (Eb7)—> Abm //
	C—> Fm—> Ab—> Eb7—> Abm //
	C—> (G7—> C—>) Eb7 (Bbm7)—> Abm
A	C—> D7—> E7—> A //
	C—> F—> G—> Am—> E7—> A
Am	C—> F—> E (E7)—> Am //
	C—> (F—> G7—> C—>) Am (—> Dm—> E7—> Am)
Bb	C—> C7—> F7—> Bb //
	C—> Cm—> F—> F7—> Bb //
	C—> G—> F—> F7—> Bb
Bbm	C—> Cmaj7—> Abmaj7—> Fm7—> F7—Bbm //
	C—> Efulldim7—> Fm7—> F7 (F7b9)—> Bam //
	C—> Eb—> Ab—> F7—> Bbm
B	C—> C/B—> Am—> F#7lDb7—> B //
	C—> Em—> F#lGb (F#7lGb7)—> Bsus4—> B
Bm	C—> Em—> A—> F#7lGb7—> Bm //
	C—> F—> G—> F#7lGb7 —> Bm //

C—> Am (C/E)—> F#s4|Gbs4—> F#|Gb—> Bm

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From Am to **Sequence of viable chord progressions for modulation**

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A	**Am—> A //**
	Am—> Dm—> D—> E7—> A //
	Am—> Dm—> E7—> A //
	Am—> F—> E (E7)—> A
Am	**Am—> Am**
Bb	**Am—> Dm—> Fm6—> F7—> Bb //**
	Am—> E7—> F7—> Bb //
	Am—> E7—> Bb //
	Am—> F—> F7—> Bb
B	**Am—> C—> D#fulldim7|Ebfulldim7 (B7)—> F#7|Gb7—> Bmaj7—> B //**
	Am—> E7—> G#7|Ab7—> C#m|Dbm—> F#7|Gb7—> B //
	Am—> Fmaj7—> F#halfdim7|Gbhalfdim7—> F#7|Gb7—> B
Bm	**Am—> C—> D—> F#7|Gb7—> Bm //**
	Am—> Dm—> G (G7)—> D (D7)—> F#7|Gb7—> Bm //
	Am—> C—> D—> F#7|Gb7—> Bm
C	**Am—> Dm—> G7—> C //**
	Am—> (F—> E7—> Am—>) C (—> F—> G—> C) //
	Am—> F—> G—> C
Cm	**Am—> Dm—> Abm6 (Fm)—> G7—> Cm //**
	Am—> Dm—> Db7—> Cm //
	Am—> F—> C—> G7—> Cm
Db	**Am—> A7 (Afulldim7)—> Bbm7—> Ebm7—> Ab7—> Dbmaj7—> Db //**
	Am—> Bbmaj7—> Fm (Fm/Ab)—> Ab7 (Abmaj7)—> Dbmaj7—> Db //
	Am—> Fmaj7/A—> Fm6/Ab—> G7 (G7/F)—> C7 (C/E)—> …
	… —> Dbmaj7—> Db
Dbm	**Am—> A7—> Dm—> Ab7—> Db—> Dbm//**
	Am—> E7—> Db7 (Abfulldim7)—> Dbm //
	Am—> F—> Ab—> Dbm
D	**Am—> B7—> E7—> A7—> D //**
	Am—> Dm—> A7—> D
Dm	**Am—> Eb—> G (G7)—> Dm**
Eb	**Am—> Afulldim7—> Bb7—> Eb //**
	Am—> A—>Eb //
	Am—> F—> Bb7—> Eb
Ebm	**Am—> Dm—> E7—> Ab(fulldim)7 —> Abm7—> Bb7—> Ebm //**
	Am—> Dm—> F—> E7—> Fm—> Bbm—> Bbm7—> Eb7—> Ab—> …
	… —> (Bb7)—> Ebm //
	Am—> F—> C—> Eb—> (Bb—>) Ebm (—>Gb—> Abm—> Bb7—> …
	… —> Ebm)
E	**Am—> A—> E //**
	Am—> Am7—> F#m7b5|Gbm7b5—> B7 —> Es4—> E //
	Am—> Bfulldim7—> E //

	Am—> F#mlGbm—> E //
	Am—> Fm—> D7—> E
Em	**Am—> Dm—> B7—> Em //**
	Am—> Fmaj7—> F#halfdim7lGbhalfdim7—> B7b9—> Em9—> Em
F	**Am—> Dm—> E7—> F (—> Gm7 C7—> F0 //**
	Am—> F—> Dm—> Bb—> C7—> F / Am—> Em—> C7—> F
Fm	**Am—> Bb/D—> E (Efulldim7)—> Fm //**
	Am—> Dm—> C—> C7—> Fm //
	Am—> F—> Ab—> C7—> Fm
F#lGb	**Am—> Am7—> D7—> Ab(dim)7—> F#maj7lGbmaj7—> F#lGb //**
	Am—> Dm—> Bm7b5—> E7—> F#mlGbm—> F#lGb //
	Am—> F—> C#lDb—> F#lGb //
	Am—> F—> (Dm)—> C#lDb (C#7lDb7, Bb[fulldim]7)—> C#7—> …
	… —> Db7—> F#lGb //
	Am—> Fmaj7—> C#7lDb7—> F#lGb
F#mlGbm	**Am—> Am11—> F#m11lGbm11—> F#mlGbm //**
	Am—> (Dm—> Am—>) C—> C#/BlDb/B—> F#m/AlGbm/A —> …
	… —> F#mlGbm—> Bm—> C#7lDb7—> F#mlGbm //
	Am—> (F—> C—> G—>) C#7lDb7 (B/C#lB/Db)—> F#mlGbm …
	… (—> D—> E—> C#7lDb7—> F#mlGbm) //
	Am—> (F—> G—>) A—> D—> E7—> F#mlGbm (—> D—> E—> …
	… —> F#mlGbm) //
G	**Am—> (Dm—> E—>Am—>) A7—> G //**
	Am—> F—> C—> D7 (Am) D—> —> G //
	Am—> (F—> E7—> Am—>) D7—> G //
	Am—> F—> Abfulldim7—> E (—> D7)—> G
Gm	**Am—> Afulldim7—> Bb—> D7—> Gm //**
	Am—> A—> Dm—> G7—> Cm—> D7—> Gm //
	Am—> Dm—> G7—> Cm—> D7—> Gm //
	Am—> F—> C —> D7—> Gm //
	Am—> F—> D7—> Gm
Ab	**Am—> Dm—> E7—> Abm7—> Db7—> Ab (—> Bbm7—> Eb7—> Ab) //**
	Am—> F—> E7—> Ab (—> Db—> Eb7—> Ab) //
	Am—> F—> E7—> Eb7b5—> Abmaj7—> Dbmaj7—> Bbm7—> …
	… —> Eb7—>Ab //
Abm	**Am—> Abm //**
	Am—> C—> Eb7—> Abm //
	Am—> (Ebm7—>) Ab7—> Abm //
	Am—> F—> Bb7—> Eb7—> Abm

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Legend.	—>	Progression = "InitialChord—> NextChord—> … —> FinalChord"
	l	Separating enharmonically equivalent notes/chords, e.g., F#lGb
	//	Separating different progressions
	/	Slash chord *(All chord inversions can be written as slash chords, but not all slash chords can be represented as chord inversions, e.g., consider a "pedal point.")*
	(…)	Alternative chord or chord progression that can be included

Note. Several chords included in these progressions used for modulation should in some cases be written correctly with a '#' or 'b' — instead of a written 'b' or '#' — to represent the proper underlying key and harmonic function. Further, the listed chords must be viewed as mere "representatives" that one must play in root position or in an inversion. *In order to achieve a specific degree of "smoothness" or desired "effect" when playing these progressions on a piano, for example, one will have to consider voice-leading principles and choosing inversions and specific notes of the chords in certain octaves.* Here, the main goal is to provide sufficient information guiding the use of these progressions — possibly after enharmonic re-interpretation and/or inversion — and playing them on a keyboard. From a theoretical perspective, one must understand the underlying harmonic principles employed by such progressions. From a practical perspective, one must eventually memorize progressions and, finally, "automatically" utilize best-possible chord fingerings when pressing the required physical piano keys. Ultimately, one must "consult" one's keyboard to explore and practice progressions that satisfy one's objectives for achieving a certain effect via modulation to new keys.

Remark. These notes are merely providing a sketch of selected aspects of harmony. They lack the generally expected scholarly rigor that should be employed when one attempts to properly expose, discuss and explain the high-lighted fundamental issues being the foundation of harmony, while addressing in detail its diverse, complex and intricate relationships. Nevertheless, it is hoped that the notes might still be helpful in some minor way or possibly stimulate others to explore the topics in more depth and with a scientific attitude and desirable meticulous precision.

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Musik ist eine höhere Offenbarung als alle Weisheit und Philosophie.
(Music is a higher revelation than all wisdom and philosophy.)
Ludwig van Beethoven
(as recorded by Bettina von Arnim in a letter to Johann Wolfgang von Goethe)

APPENDIX: RELATIONSHIPS BETWEEN MUSIC THEORY AND GEOMETRIC TRANSFORMATIONS, LINEAR ALGEBRA AND BEST APPROXIMATION

The thoughts sketched here are not refined, not discussed in detail, not supplemented with in-depth examples for clarification and not supported with a generally applicable theory. The ideas in this Appendix have not been confirmed in practice. The intention is merely to stimulate a person's interest in exploring other possibilities for representing, analyzing and defining complex progressions and modulations via untraditional means. For this purpose, the Appendix points out a few techniques well-known in mathematics (geometry, algebra and approximation) and potential ways to adapt them to the actual requirements of the creation of progressions and modulations. Substantial work would have to be done if one wanted to develop a sound and musically meaningful framework for a **numerical approach to harmony** based on some of the ideas expressed here.

Most importantly, if one intend to utilize numerical methods for the representation and manipulation of notes, scales, chords etc., one will have d**efine an "equivalence" between the 12 tones of the chromatic scale and the integers 0, 1, 2, …, 11** (keeping octave-periodicity and modulo-12 arithmetic in mind as a consequence of repeating multiple octaves, each consisting of 12 half-tone). One can simply represent notes numerically with equivalent integers as follows:

C = 0,	**C#|Db = 1,**	**D = 2,**	**D#|Eb = 3,**	**E = 4,**	**F = 5,**
F#|Gb = 6,	**G = 7,**	**G#|Ab = 8,**	**A = 9,**	**A#|Bb = 10,**	**B = 11.**

A key's associated six triads and three 7th chords consisting purely of the seven notes of a chosen key, e.g., C-major, can be written as row-tuples, triples and quadruples:

Triads:
C = (C, E, G) = (0, 4, 7), Dm = (D, F, A) = (2, 5, 9), Em = (E, G, B) = (4, 7, 11),
F = (F, A, C) = (5, 9, 12[0]), G = (G, B, D) = (7, 11, 14[2]),
Am = (A, C, E) = (9, 12[0], 16[4]), B0 = (B, D, F) = (11, 14[2], 17[5]).
7th chords:
Cmaj7 = (C, E, G, B) = (0, 4, 7, 11), Em7 = (E, G, B, D) = (4, 7, 11, 14[2]),
Am7 = (A, C, E, G) = (9, 12[0], 16[4], 19[7]).

A scale, e.g., C-major (and its "inversions", church modes), can be written as a 7 x 7 matrix M, e.g., using row-notation:

$$
M = \begin{matrix}
C & D & E & F & G & A & B \\
D & E & F & G & A & B & C \\
\dots & & & & & & \\
F & G & A & B & C & D & E \\
\dots & & & & & & \\
A & B & C & D & E & F & G \\
B & C & D & E & F & G & A
\end{matrix}
=
\begin{matrix}
0 & 2 & 4 & 5 & \dots \\
2 & 4 & 5 & 7 & \dots \\
\dots & & & & \dots \\
5 & 7 & 9 & 11 & \dots \\
\dots & & & & \dots \\
9 & 11 & 12[0] & 14[2] & \dots \\
11 & 12 & 14[2] & 16[4] & \dots
\end{matrix}
$$

One can view a modulation from C-major to G-major, for example, as a map defined by an image or result matrix M':

	G	A	B	C	D	E	F#IGb	7	9	11	12[0]	…
	A	B	C	D	E	F#IGb	G	9	11	12[0]	14[2]	…
	…							…				…
M' =	C	D	E	F#IGb	G	A	B	= 12[0]	14[0]	16[4]	18[6]	…
	…							…				…
	E	F#IGb	G	A	B	C	D	16[4]	18[6]	19[7]	21[9]	…
	F#IGb	G	A	B	C	D	E	18[6]	19[7]	21[9]	23[11]	…

By using the seven notes of C-major and G-major in such a representation, a mapping, i.e., a progression/modulation from C-major to G-major, can be defined by a matrix T, a 7 x 7 transformation matrix, defining the transformation T x M = M'. (One must keep in mind that T can be a matrix with real numbers as entries, while M and M' have only integer values as entries since they represent discrete notes.) When modulating from C-major to G-major, only the notes G and F#IGb are not shared by the two keys. One can adopt two perspectives: (1) An original key with its associated scale is given as a matrix M, a transformation matrix T is provided as well and their matrix product T x M determines the matrix M', i.e., the new notes and scales. (2) An original key with its associated scales is given as a matrix M, a new key and its scales are provided as an image matrix M' and a transformation matrix T must be calculated such that T x M is equal to the matrix M'.

A general key change from C-major to G-major can also be represented by the seven triad changes resulting from such a modulation:

Triad changes:
C = (C, E, G) = (0, 4, 7) —> G = (G, B, D) = (7, 11, 14[2]) and
Dm—> Am, Em—> Bm, F—> C, G—> D, Am—> Em, B0—> F#0IGb0.

The seven scale-associated triads of C- and G-major can be written as 7 x 3 matrices (7 x 3: 7 rows, 3 columns) with rows representing the respective integer tuples (triples) of the triads, integer values relative to the "origin note" C = 0 and all values to be understood as modulo-12 values:

		C	E	G		0	4	7
		D	F	A		2	5	9
		E	G	B		4	7	11
C_triads	=	F	A	C	=	5	9	12
		G	B	D		7	11	14
		A	C	E		9	12	16
		B	D	F		11	14	17

		G	B	D		7	11	14
		A	C	E		9	12	16
		B	D	F#IGb		11	14	18
G_triads	=	C	E	G	=	12	16	19
		D	F#IGb	A		14	18	21
		E	G	B		16	19	23
		F#IGb	A	C		18	21	24

In geometry, basic geometrical transformations are linear transformations. These transformations are **translation, scaling, reflection and rotation (performed relative to a local origin, a "center")**. By representing a scale or a chord as a tuple, i.e., a row — analogously to representing a point's coordinates in a space by a coordinate tuple — one can i**nterpret progressions and modulations as linear transformations**. Keys, scales, chords, progressions, modulations etc. could therefore be represented and/or performed via **numerical calculations**, using linear algebra and matrix algebra.

Elementary **geometrical linear transformations are fundamentally based on the use of linear algebra notation and operations**. Transformation matrices can be applied to geometrical data, e.g., to points whose locations/positional vectors are written as rows of coordinate tuples. The basic transformations translation, scaling, reflection and rotation are generally performed relative to a local origin, a center, i.e., a local center point). A rotation around a center point in the plane, for example, can also be represented by two reflections defined by two reflection axes passing through the center, with these two reflections performed in sequence, i.e., by a concatenation of two reflections. As an anecdote, Johann Sebastian Bach's counterpoint and canon compositions exhibit underlying principles of geometrical transformations, but Johann Sebastian Bach might not have utilized strict geometrical transformation principles directly in his compositional practice, and only later analyses have determined that some of his compositions could be viewed as implementations of geometrical ideas (e.g., Bach's "Crab Canon").

Abstractly, **a new, result chord-matrix ChordsNew can be obtained by multiplying two matrices and adding a "translation-note-tuple" t as follows: ChordsNew = T x ChordsOld + t**. (As done in geometry, one can use so-called homogeneous coordinates as representation — implying the use of a matrix Thomog, i.e., T with an added bottom row, the "translation-note-tuple" row) — which allows one to perform the entire transformation into a single transformation requiring only one matrix-times-matrix operation. Since a rotation in the plane around a center can be described by two concatenated reflections, the **needed** basic geometrical transformations are **only translation, scaling and reflection**. **Given a local "center note," translating a set of notes means shifting all notes upward or downward by the same amount of half-steps/steps; scaling a set of notes means expanding or compressing the distribution of notes leading to larger or smaller step-distances between notes; and reflecting a set of notes means "mirroring" all notes to the other side of the chosen "center note**."

In other words: A matrix M', assuming row notation, contains as rows the "images" of the original row representation of original (sets of) notes. One can either define a matrix T and a tuple t (or Thomog) and its entries to determine new notes, chords, scales etc., or one can determine the unknown entries of a transformation matrix T and the tuple t (or Thomog) by using a given set of original and new, transformed row representations of notes, chords, scales etc. (One must keep in mind that both under- and over-determined cases will arise, i.e., cases where not enough conditions are given to determine one unique transformation and cases where to many conditions are given and one must determine a transformation that satisfies the conditions in a "best-possible" way.)

One can **adopt such a geometrical viewpoint** for the representation and/or execution of key changes, progressions, modulations etc. In the musical setting, one can think of a key, a scale, a chord etc. as a point or positional vector defined by a set of notes (usually ordered from lowest to highest note). A linear mapping of scales or chords, all represented as rows of a matrix, can abstractly be thought of as mapping an entire set of scales or chords (defining matrix rows) to a new image matrix, containing the new notes of the scales or chords that result, i.e., **[newChords] = [Thomog] x [oldChords]**. Here, the notation "[…]" is used to make clear that the transformation concerns an entire set of scales or chords. When using a homogeneous representation, the "translation-note-tuple" t with an "appended 0" as its last coordinate defines the last row of the transformation matrix Thomog, and an "unnecessary 1" coordinate value must be included as last coordinate value of every row in the Mhomog matrix.

One can also combine or **concatenate multiple transformations defined by their respective homogeneous matrices** Th1, Th2, …, Thk to represent more complex progressions or modulations, where one transitions from an intermediate scale or set of chords to another intermediate scale or set of chords, until a final destination is reached, represented by M'homog. In this case, a concatenation of k transformations would be represented as the product

$$
\begin{aligned}
M'homog &= [Thk] \times (\dots \times ([Th2] \times ([Th1] \times [Mhomog])) \dots) \\
&= [Thk] \times (\dots \times ([Th2] \times Mhomog_intermediate1) \dots) \\
&= [Thk] \times (\dots \times Mhomog_intermediate2 \dots) \\
&= \dots \\
&= M'homog.
\end{aligned}
$$

Distance is a fundamental concept one must consider and use when thinking about similarity of keys, scales, chords, sets of chords etc. For example, the degree of smoothness (or lack thereof) of a progression of chords in a key modulation depends on the **definition of distance in the context of harmony and harmony-related, important aspects**. The **circle-of-fifths-order**, for example, orders keys — i.e., defines an ordered set of keys — in such a way that each pair of neighboring keys on the circle has minimal distance with respect to the keys' associated notes: Neighboring keys differ by only one note.

Consequentially, the numbers of common triads and 7th chords of neighboring keys, using only the notes of neighboring keys, are also maximal.

For example, the C-major triads are C, Dm, Em, F, G, Am, B0; its 7th chords are Cmaj7, Em7 and Am7, consisting entirely of notes of the C-major scale. G-major triads are G, Am, Bm, C, D, Em, F#IGb0; its 7th chords are Cmaj7, Em7 and Am7, using only notes belonging to the G-major scale. Thus, C-major and G-major share four triads and three 7th chords: C, Em, G and Am and Cmaj7, Em7 and Am7. These **keys have "minimal distance" and "maximal similarity."** It is possible to perform smooth modulations between them via shared chords, their "pivot chords." Analogously, C-major also shares four triads and three 7th chords with its "left neighbor" on the circle, F-major. Further, D-major's triads and 7th chords are D, Em, F#IGbm, G, A, Bm, C#0IDb0 and Dmaj7, Em7 and Bm7, respectively. Thus, C-major and D-major only have three triads and one 7th chord in common. When moving away from C-major even more on the circle, e.g., to A-major with triads A, Bm, C#IDbm, D, E, F#mIGbm, G#0IAb0, no triads or 7th chords exist that are shared between with C- and A-major. If one established a matrix with 12 rows and 12 columns, one row and one column for each key, using the circle-of-fifths-order for rows and columns, one can list the triads and 7th chords shared for each key pair in a specific matrix position (and also the notes that are not shared).

If one wants to compute a **best approximation with respect to optimizing distance (minimizing distance)** between an original and a transformed set of notes, keys, scales, chords etc, one will have to define and involve a proper measure of distance — in the context of constructing smooth modulation or chord sequences, for example. For musical progressions and modulations, one would have to define **distance such that it is harmonically (or in some other musically relevant sense) meaningful**. The circle-of-fifths-order defines an order for all keys, ordering the 12 keys in a musically, harmonically meaningful way. Adapting a best approximation approach for the construction of a "good," "as-smooth-as-possible" chord-to-chord progression or modulation involving rather different original and destination keys might be a helpful if one properly adapts known numerical linear algebra methods. Best approximation methods usually minimize distance in an average sense, a "least-squares" sense, where a best transformation, best approximation "deforms," original data only minimally to generate the final data, i.e., the final destination of a progression. This aspect of **best approximation could potentially be valuable in the musical setting**.