8/20/2024

Chords and Transitioning between Different Keys —

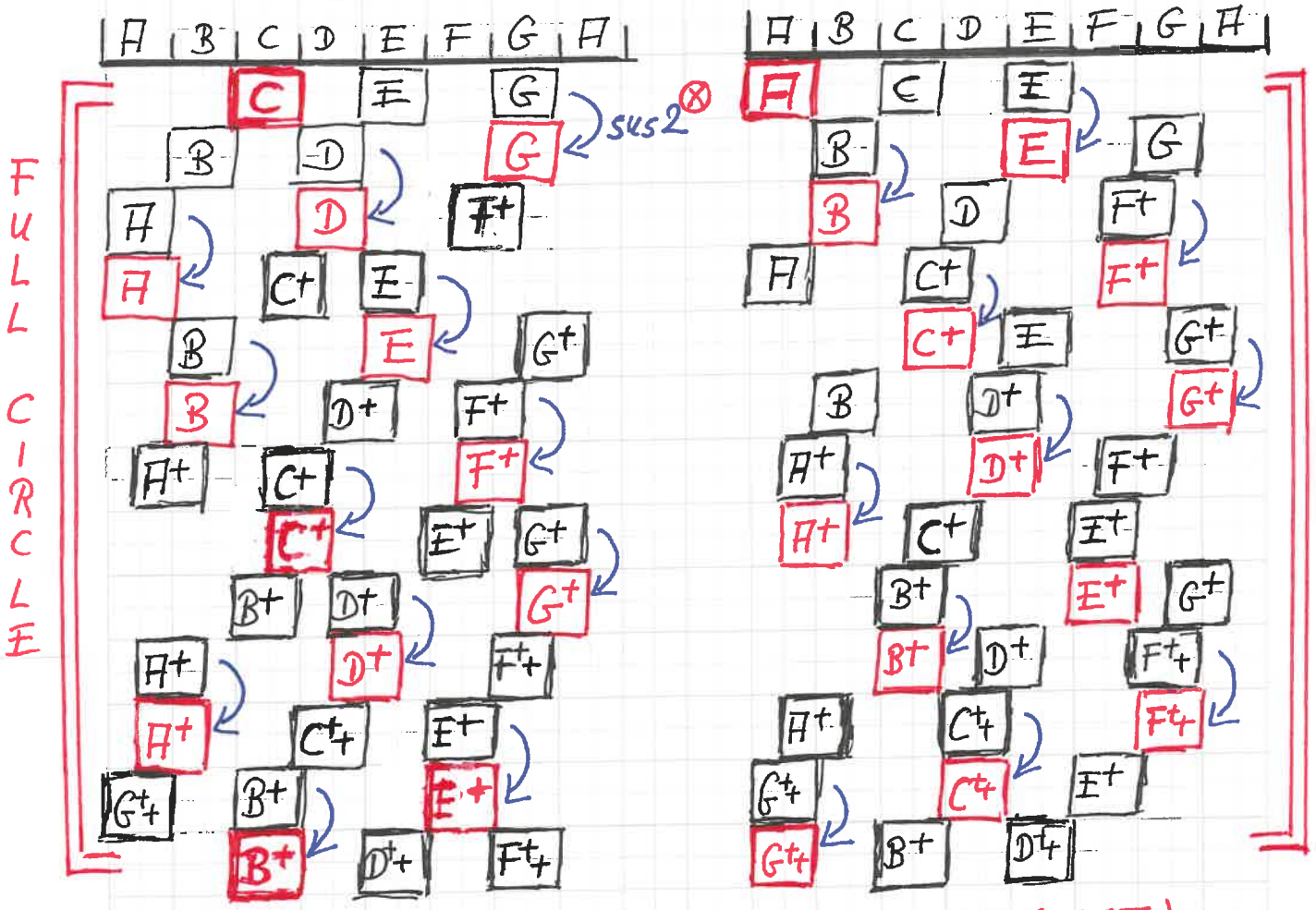
Chords in a Specific Key (Major and Minor Chords) / "Transition Chords" / Circle of 5ths:

- **Key of C (triads):** CEG, DFA, EGB, FAC, GBD, ACE, BDF (+ inversions)
- **Chords with root C:** CEG (maj), CE^bG (min), CDG (sus2), CFG (sus4), CE^bG^b (dim), $CEG^\#$ (aug.)
- **"Transitions":**
  1. CEG → CDG → BDG ; (sus2 → G maj chord)
  2. CEG → CFG → CFA ; (sus4 → F maj chord)
  3. CEG → $CEG^\#$ → CEA ; (aug. → A min chord) ...
- **Circle of 5ths:** ... B^b ↔ F ↔ C ↔ G ↔ D ↔ ... MAJ.
  - IV: GBD → $GBD^\#$ (aug.) → GBE (E min chord) ; (sus4)
  - V: FAC → $FAC^\#$ (aug.) → FAD (D min chord)
  - ... g ↔ d ↔ a ↔ e ↔ b ↔ ... MIN.
  - i: ACE → ABE (sus2) → GBE (E min chord) ; (sus4)
  - ii: ACE → ADE → ADF (D min chord)
  - 3: CEG → $CEG^\#$ (aug.) → CEA (A min chord) ; ...

Tonika – Dominant – Subdominant – Nat. Minor Relationships

⇒ CAN USE "TRANSITION CHORDS" TO "SMOOTLY MOVE" FROM CHORDS OF ANY KEY TO ALL CHORDS OF THE KEYS IN THE "CIRCLE-OF-5TH's NEIGHBORHOOD"!

- **MODULATION – Moving from Chord to Chord / "Harmonic Distance" between Key Signatures & Chords etc.** 8/21/2024



($F^{+} \hat{=} F^{\#}$ etc.; $E^{+} \hat{=} F$, $B^{+} \hat{=} C$; $F^{+}_{+} \hat{=} G$, $C^{+}_{+} \hat{=} D$, $G^{+}_{+} \hat{=} A$, $D^{+}_{+} \hat{=} E$)

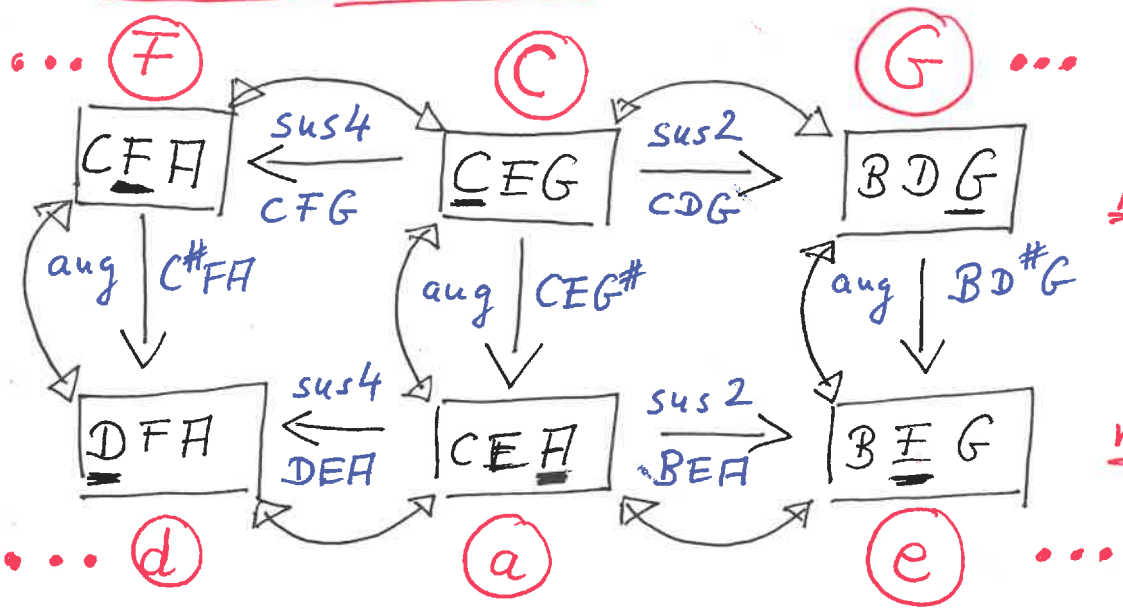
- Tables list the possible 12 major and 12 minor triads using the order of the circle of 5ths, traversed in clockwise order for a full cycle.
- Since the tables show triads only in the interval/range [A, B, C, ..., G^{+}], first and second inversions are included.
- "Harmonically plausible" forward (key) progressions in clockwise circle-of-5ths order are shown via ↘, e.g.,:
⊗ CEG $\xrightarrow[sus2]{}$ CDG → BDG (from C to G major).

MODULATION – cont'd.

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(for "permanent" or "temporary" change of key signature)

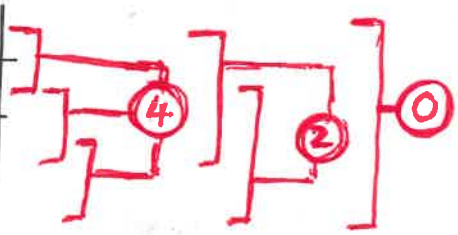
→ Smooth modulation in a "6-key neighborhood" of circle of 5th's:



→ Use sus2 (sus4) chord to move clockwise (counter-clockwise) from MAJOR (minor) to MAJOR (minor); use aug. chord to move from MAJOR to (natural) minor.

• Use common chords (triads) for modulations:

	I	ii	iii	IV	V	vi	DIM
C	CEG	DFA	EGB	FAC	GBD	ACE	BDF
G	GBD	ACE	BDF#	CEG	DF#A	EGB	F#AC
D	DF#A	EGB	F#AC#	GBD	AC#E	BDF#	C#EG
A	AC#E	BDF#	C#EG#	DF#A	EG#B	F#AC#	G#BD
...	...						



⇒ Neighbor keys (in the circle of 5th's) have 4 triads in common; "neighbor-of-neighbor keys" have 2 triads in common.

C	E	G		(135)
C		G	E	(153)
	E	G	C	(351)
(315)	E		C	G
(513)		G	C	E
(531)		G	E	C

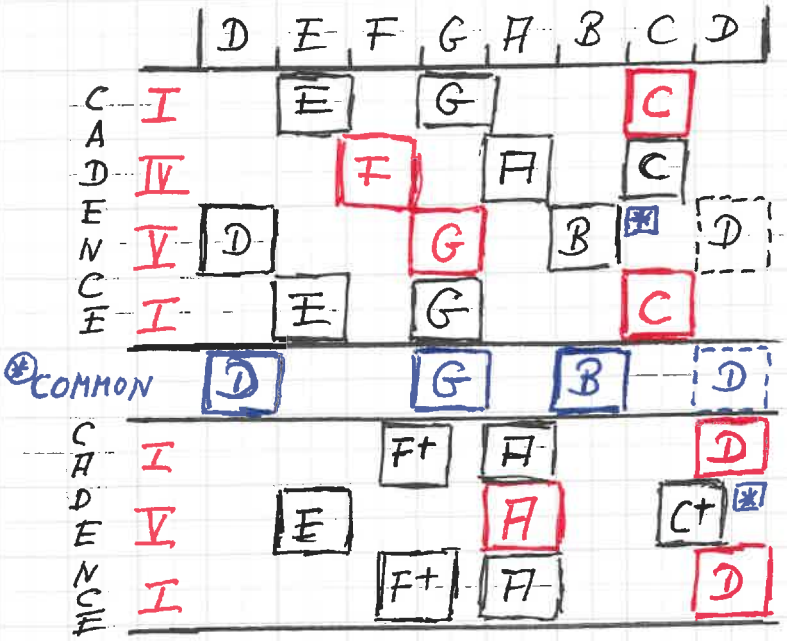
Note. There exist 6 = 3! possible triads of three different notes. Thus, the "true number" of possible (in-) versions is 6 – and one must consider TWO OCTAVES for their "implementation." SEE LEFT FIGURE.

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• MODULATION: Moving from the Current Key (Signature) to a New Key

[Ref.: Franz TITSCHER, pianobeat.de, "Modulationen..."]

- Standard / General Modulation:
  - i) CADENCE for current key, e.g., I-IV-V-I
  - ii) Determine all CHORDS that the current Key and new key have IN COMMON (triads).
    - Key of C (current): CEG, DFA, EGB, FAC, GBD, ACE
    - Key of D: DF#A, EGB, F#AC#, GBD, AC#E, C#EG
    - → Common triads: EGB, GBD
  - iii) Pick one of the common chords to modulate! (transition / modulation chord)
  - iv) CADENCE for new key, e.g., I-IV-V-I or I-V-I



- Example: from C to D (from EGC to F#AD):
  - Cadence for C: EGC → FAC → DGB(D) → EGC
  - Transition / common chord: DGB(D)
  - Cadence for D: F#AD → EAC# → F#AD
- Other transitions / transition chords & cadences possible!

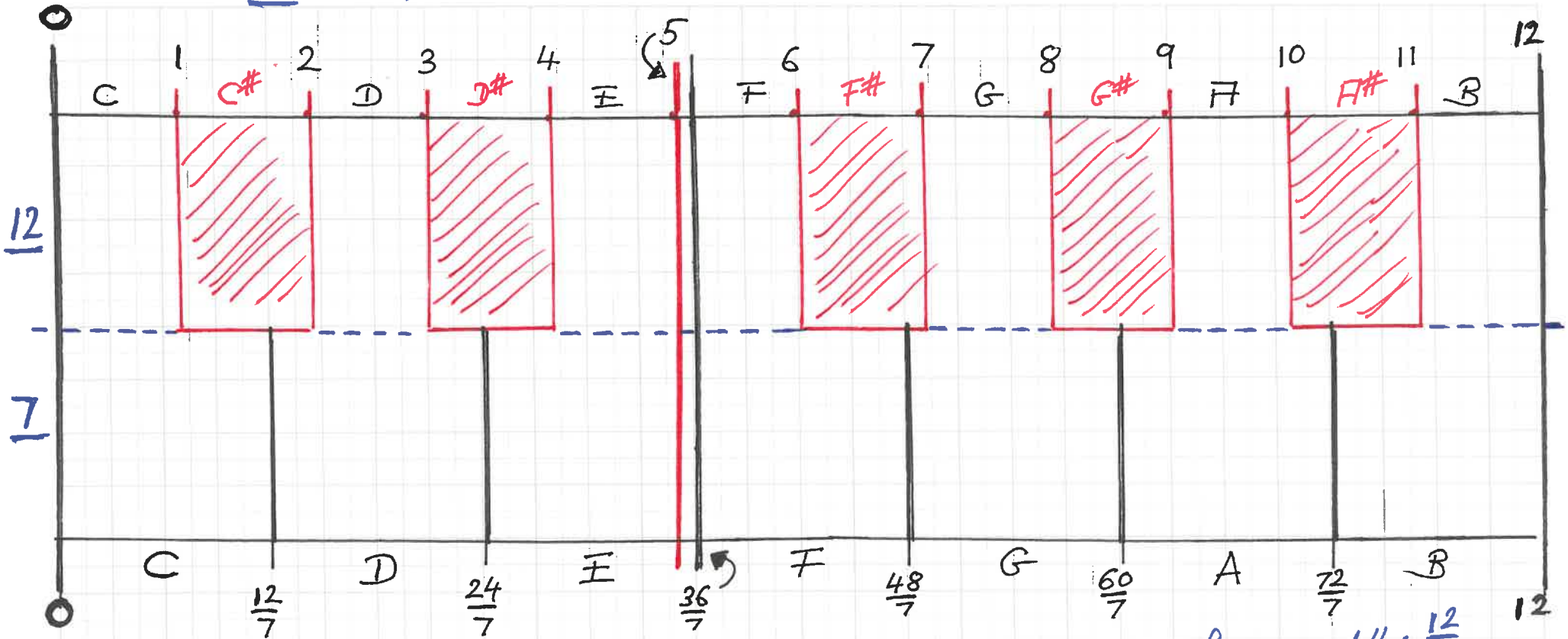
* Common chord = "Bindeakkord" ("connecting" / "glue chord")

* B = Lead note to C, C# = Lead note to D ⇒ USE CHORDS WITH Lead notes effectively: moving up by ½ note!

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LAYOUT OF AN OCTAVE

12 half-tones — uniform spacing ⇒ uniform width: 1



7 tones/notes of C — uniform spacing ⇒ uniform width: 12/7

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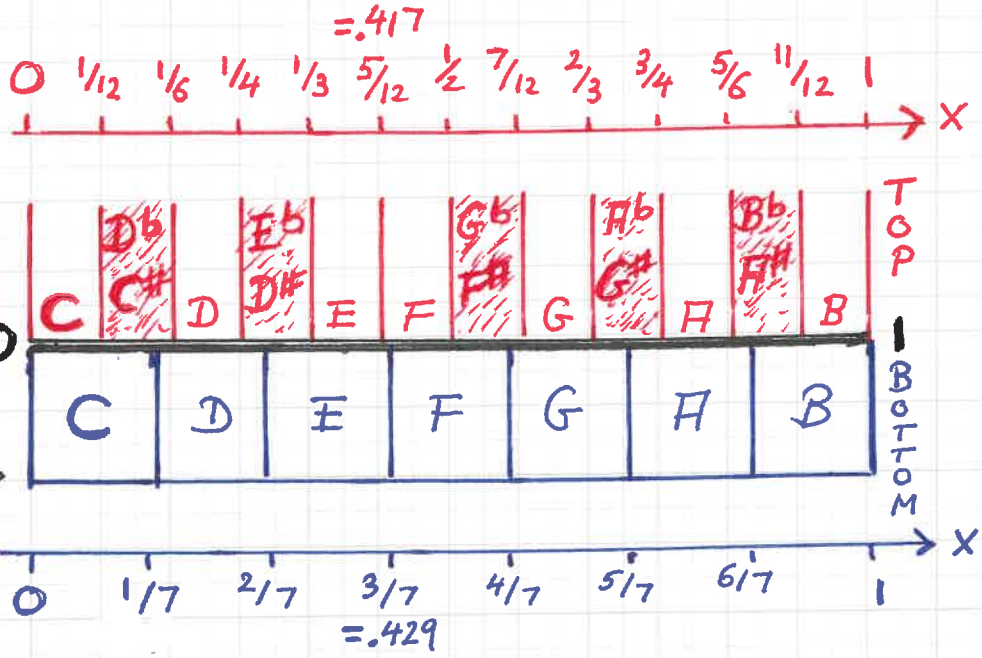
• Note. The physical placement (and approximate width) of keys should "ideally" be based on a UNIFORM spacing and width of the physical keys. Why? Chords and scales must be "remembered" i.e. "shapes" should be "invariant" under a change of key signature. Of course, "shapes" should be "memorized". Ideally, "shapes" should be "invariant" under a change of key signature. Of course an "OPTIMAL LAYOUT" is (still) an "unsolved problem." ~

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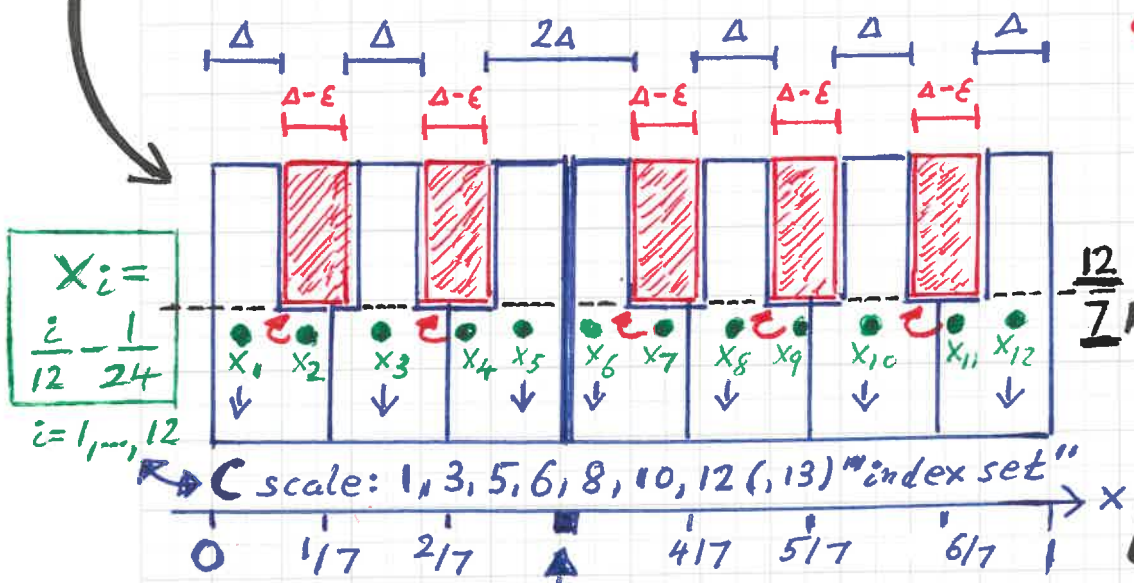
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Note. Physical Layout of Piano Keyboard – Key Placement & Width

8/25/2024



- The keyboard has a "TOP" & "BOTTOM" part.
  "TOP": 12-note chromatic scale;
  "BOTTOM": 7 notes of "natural octave"



- FINGERS MUST FIT BETWEEN BLACK KEYS: SPECIFY ε!

$\Rightarrow 7\Delta + 5(\Delta - \varepsilon) = 1$

$\Rightarrow \Delta = \frac{1}{12}(1 + 5\varepsilon)$

(e.g., $\varepsilon = 1/100 \Rightarrow \Delta = 7/80$)

- At the "INTERFACE" between the 12-key & 7-key parts, SHAPES OF ALL CHORDS can be understood and memorized as SAME SHAPES REGARDLESS OF KEY SIGNATURE! This fact is a consequence of the EQUIDISTANT SPACING BETWEEN THE LOCATIONS/POINTS "•", $X_1 \ldots X_{12}$.

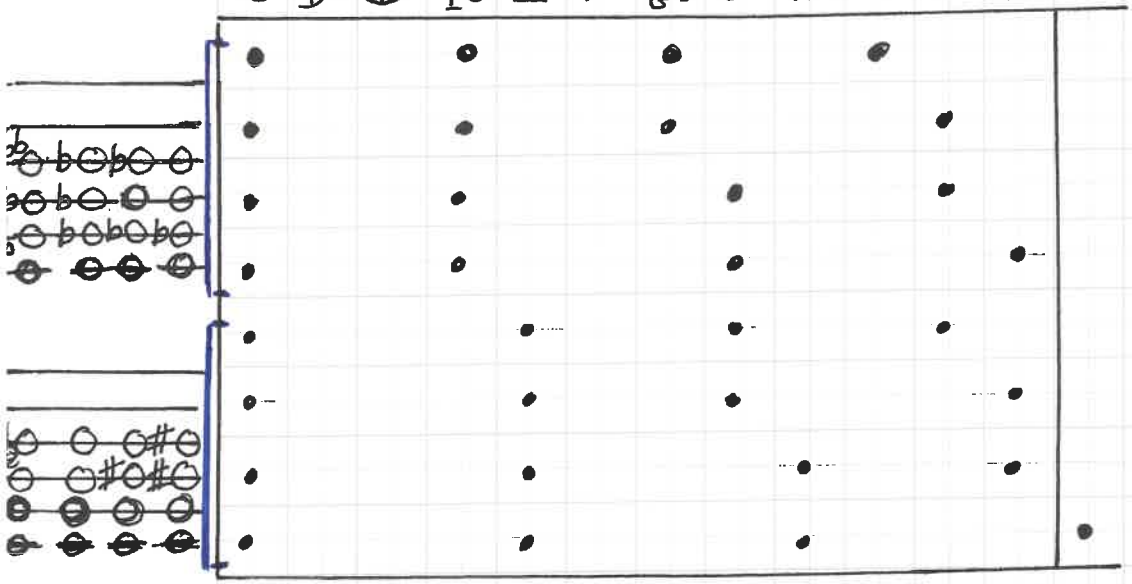
INDICES all maj. (min.) SCALES "starting on X_i":
$\{i, i+2, i+4, i+5, i+7, i+9, i+11 (, i+12), \{i, i+2, i+3, i+5, i+7, i+8, i+10 (, i+12)\}\}$.

• STACKING 3rds

8/26/2024

(1) —— half-tones —— (12)

C Db D Eb E F Gb G Ab A Bb B C



C Eb Gb A (=Bbb)
C Eb Gb Bb
C Eb G Bb
C Eb G B
C E G Bb
C E G B (=G#)
C E Ab B
C E Ab C (=B#)

8 = 2³ possibilities to stack 3 minor and major 3rds on top of each other; root = C

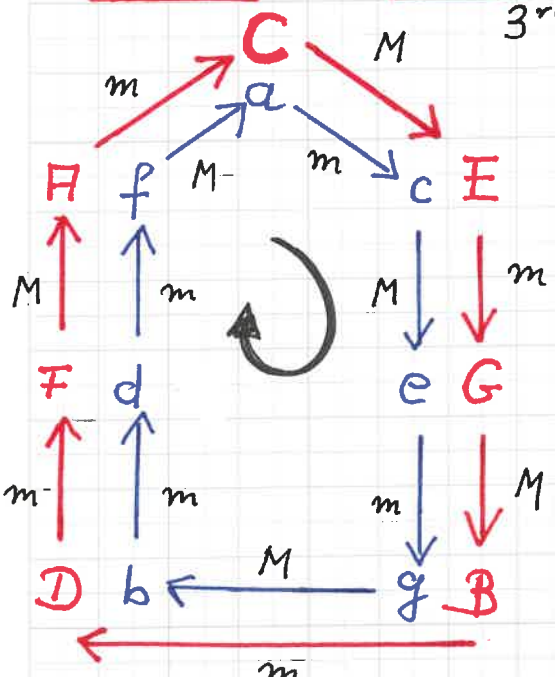
(PLUS: INVERSIONS!)

→ The specific chords resulting from stacking 3 3rds exhibit varying degrees of "consonance." (For example, CEAbC is "unpleasant.")

→ The circle of 3rds is not commonly known/used. It illustrates how one can stack minor (m) and major (M) 3rds — i.e., stacking 7 3rds to cover two octaves, for major and minor key signatures (CEGBDFA(c), acegbdf(a)).

• "CIRCLE OF 3rds"

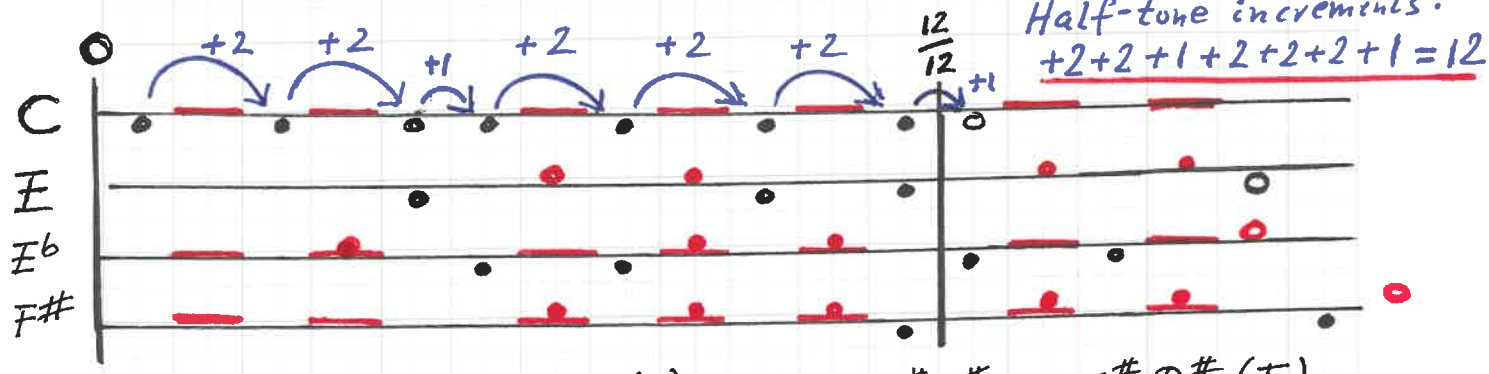
(m = minor 3rd, M = major 3rd)



CEGBDFA(C) major & acegbdf(a) minor cirles by stacking 3rds

"Key Signature & Chromatic Notation" 8/26/2024

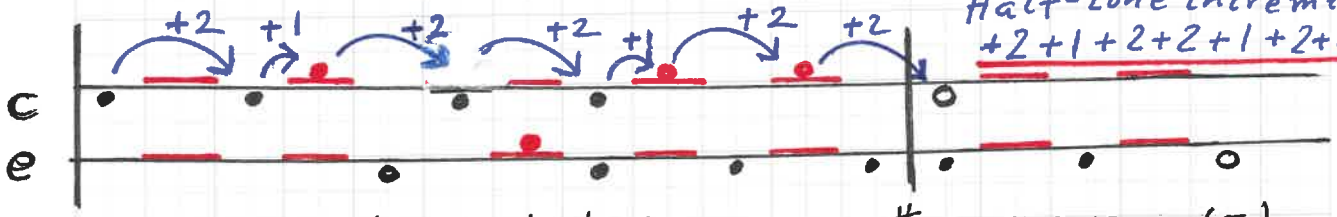
C C# D D# E F F# G G# A A# B C C# D D# E F F#



Half-tone increments: +2+2+1+2+2+2+1 = 12

(MAJ.) C: C D E F G A B (C) E: E F# G# A B C# D# (E)

E♭: E♭ F G A♭ B♭ C D (E♭) F#: F# G# A# B C# D# E# (F#)

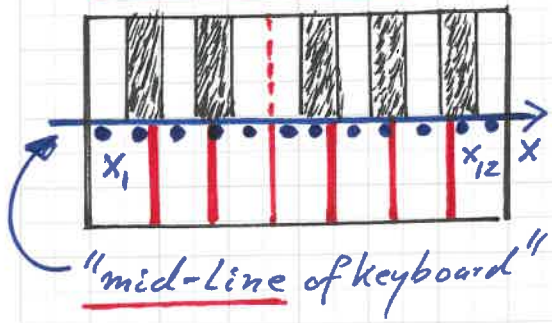


Half-tone increments: +2+1+2+2+1+2+2 = 12

(MIN.) c: C D E♭ F G A♭ B♭ (C) e: E F# G A B C D (E)

→ Examples of major & minor scales shown in the context of chromatic notation / a "chromatic keyboard" / the "top part of the piano keyboard — where half-tones have the SAME, UNIFORM EQUIDISTANT SPACING/POSITIONING!

→ IMPLICATION for "SHAPES & FINGERING":



{•} = {x1, ..., x12} = set of equidistant midpoints of the "top chromatic keys of same width"

→ TIPS OF FINGERS: HOLD THEM AS CLOSELY AS POSSIBLE TO THE xi-VALUES ON THE x-LINE; "MOVE THE HAND INWARDS / UPWARDS — WITH THE TIP OF THE THUMB AS CLOSE AS POSSIBLE TO THE "MID-LINE"!

!!! ⇒ CHORD SHAPES & FINGERINGS EQUAL FOR ALL KEYS!!!

• More on Chords — Chord Distances and Their Visualization

8/27/2024

→ Motivating examples for distance definition:
(using integers after note names to indicate octave on keyboard, e.g., C1 ... C8)

dist(C1, C1) = 0 ; dist(C3, C4) = ? small ? 0 ? ;

dist(C3, C4) ≤ dist(C3, C5) ;

dist((C,E,G), (C,E,G)) = 0 ; dist((C,E,G), (E,G,C)) = ? ;

dist((C1,E1,G1), (C2,E2,G2)) ≤ dist((C1,E1,G1), (C3,E3,G3)) ;

dist((C,G), (C,E,G)) = ? 0 ? small ?? ;

dist((C1,G1), (C2,E2,A2)) = ???

VIS. ⇓

= combine(dist(C1, (C2,E2,A2)),
dist(G1, (C2,E2,A2)),
dist(C2, (C1,G1)),
dist(E2, (C1,G1)),
dist(A2, (C1,G1)) ??? ;

- (n1, n2, n3 (n4)) require 3 (4) D vis. grid ; (Δ12, Δ23 (, Δ34)) require only 2 (3) D vis. grid.
- CLOSE CHORDS OR Δ-TUPLES SHOULD BE CLOSE IN VIS. LAYOUT.

dist(c1, c2) = no of Common Notes / (len(c1) + len(c2)),
where len( ) = no of Notes in Chord( ) ;

Δ(C4, E4, A4) = combine(Δ12, Δ13, Δ23) (Δ13 = Δ12 + Δ23)
? = combine(dist(C4,E4), dist(C4,A4), dist(E4,A4))
? = combine(dist(C4,E4), dist(E4,A4))
= Δ123 = Δ13 (C4, E4, A4) = ... ???
= (Δ12, Δ23) ⇒ from triple (C4,E4,A4) to (Δ12, Δ23)

Δ(C4,E4,G4,A4) = (Δ12, Δ23, Δ34) ⇒ DIMENSION REDUCTION BY 1 .

• More on Chords — Indexing, Numerical Values, Distances and Visualization

8/27/2024

→ How should one index notes / keys "meaningfully"?

→ How should one define numerical values, to be able to define and calcute "meaningful chord distances," when associating a number with a note? "Meaningful" note-to-note distances must also lead to "meaningful" chord-to-chord distances! (Special cases: distance def. for chord inversions / permutations)

	C	C#/Db	D	D#/Eb	E	F	F#/Gb	G	G#/Ab	A
half-tone steps	0	1	2	3	4	5	6	7	8	9
integer index+1	1	2	3	4	5	6	7	8	9	10
real nos. for steps	0	1/2	2	3/2	2	5/2	3	7/2	4	9/2
chord numbers	1	...	2	...	3	4	...	5	...	6
Roman numerals	I		ii		iii	IV		V		vi
...	...									:

Possible "index sets" of C.

→ ISSUES: — Notes are close in terms of lexicographical distance of notes' names (dist(C,D) < dist(C,G)), but are "harmonically very distant" (i.e., dissonant), and vice versa.

— Chords (n1, n2, n3) & (n1, n2, n3) should / must have a distance that is zero (?) or very small when they represent inversions or permutations: dist((C,E,G),(G,C,E)) ≙ "small"; dist((C1,E1,G1),(C2,E2,G2))!!

— One must define a distance for 2 chords consisting of different numbers of notes: dist((n1,n2),(n1,n2,n3)) = ?

— Distance for the notes defining ONE chord to be defined: dist(n1,n2,n3) ≙ combination(dist(n1,n2), dist(n1,n3), dist(n2,n3))

⇒ dist(c1, c2) = combination(dist(allNotes of c1 to allNotes of c2), (" c2 to " c1))
(chord, chord)

• More on Chords — "Distance"

8/27/2024

→ One can define a chord "key-specific" by providing its notes or, more generally, by providing the "distances," i.e., the number of half-tones," between pairs of two consecutive notes (in any key). For example, the specific-note triad $n1n2n3$ is more generally defined — key independently — by the expression $\Delta_{12}\Delta_{23}$; here Δ_{12} (Δ_{23}) represents the distance between $n1$ and $n2$ ($n2$ and $n3$).

Example: $n1n2n3 = CEG \Rightarrow \Delta_{12} = dist(C,E)$, $\Delta_{23} = dist(E,G)$; i.e., $\Delta_{12}\Delta_{23}$ = "combined distance expression for a general "stack of major and minor 3rd's".

$\Rightarrow$ A 3-note (4-note) chord representation $n1n2n3$ ($n1n2n3n4$) has a general 2-delta (3-delta) expression $\Delta_{12}\Delta_{23}$ ($\Delta_{12}\Delta_{23}\Delta_{34}$). THUS ONE OBTAINS A "DIMENSION REDUCTION BY 1."

→ How does this "Δ-concept" relate to the definition of distance of two chords?

Problem: chord 1 $= (n1, n2, n3) = c_1$; $c_2 = (\overline{n1}, \overline{n2}, \overline{n3})$ = chord 2

$\Rightarrow dist(c_1, c_2) = dist((n1,n2,n3), (\overline{n1}, \overline{n2}, \overline{n3})) = ?$

Problem: A def. of distance must consider that 2 chords that are inversions "are the same chord" (really?) and should have dist. zero (really?)

• More on Chords: "Chord Distances", "Chord Similarities," "Chord Inversions & Permutations," "Chord Visualization" 8/27/2024

→ **GOAL:** Definition of "chord distance" — allowing one to identify close chords, for "smooth transitions," and to visualize them

→ **Chord Inversions & Permutations:**

A 3-note (4-note) chord — e.g., triads and 7th chords — can be written as a tuple $(n1, n2, n3)$ $((n1, n2, n3, n4))$, with $n1, n2, n3$ $(n4)$ being notes of the chord. There are $6 = 3!$ permutations of a 3-note chord: $(n1,n2,n3)$, $(n1,n3,n2)$, $(n2,n1,n3)$, $(n2,n3,n1)$, $(n3,n1,n2)$, $(n3,n2,n1)$ — e.g., (CEG)*, (CGE), (ECG), (EGC)*, (GCE)*, (GEC); using 2 octaves, one can play:



* standard chord inversions.

(Analogously, there are $24 = 4!$ 4-note chord permutations: $(n1,n2,n3,n4), \ldots, (n4,n3,n2,n1)$.)

→ IF ONE DEFINES THE (numerical?) DISTANCE BETWEEN 2 NOTES $n1$ AND $\overline{n1}$ "WELL," THEN ONE WILL BE ABLE TO DEFINE AND CALCULATE CHORD DISTANCE OF 2 CHORDS $(n1, n2, n3\,(, n4))$ AND $(\overline{n1}, \overline{n2}, \overline{n3}\,(, \overline{n4}))$.

More on Chords — 7th Chords etc.

8/27/2024

→ Ref: https://musikanalyse.net/tutorials/septakkorde (in German)

→ The seven "inherent" 7th chords associated with a specific key (signature) can be defined similarly to the triads of a key.
Example: 7 7th chords of the C major key:
CEGB, DFAC, EGBD, FACE, GBDF, ACEG, BDFA.
When considering the nature of the first 3rd of these seven 7th chords, one could once again use the Roman numerals I, ii, iii, IV, V, vi, dim.
Each of these 7th chords has 3 inversions, e.g.,
CEGB (root pos.) → EGBC (1st) → GBCE (2nd) → BCEG (3rd).
(There are 7th chords for all major and minor keys, of course.)

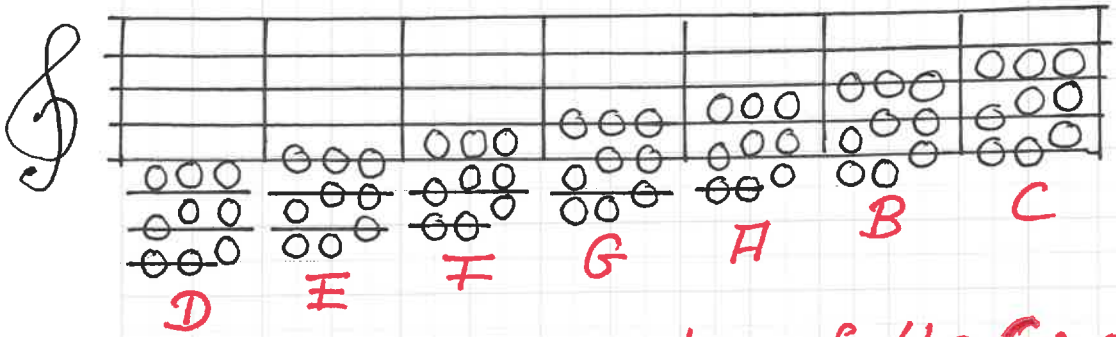
→ Purpose/use of 7th chords? Transitions, i.e., moving from one chord to another; providing tension via dissonant nature; providing a means to modulate, i.e., establish a new key signature etc. (A 7th chord used as a transition chord must/should have a note (notes) in common with a chord/key one is coming from and the chord/key one is moving to.)

→ Stacking more than 3 3rds leads to even more complicated chords/harmonies: 9th, 11th, 13th chords.

• More on CHORDS & Inversions: Triads & 4-Tones (7th Chords)

8/27/2024

→ "The highest note in a chord (played with the right hand) is the note 'being heard,' defining the 'melody'."



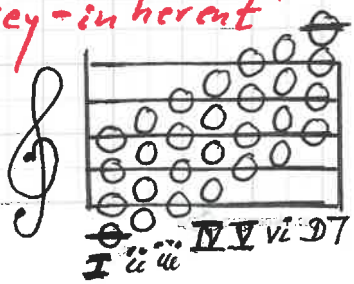
• Example: Three triads (in the key of C major) for the seven notes of the C major scale are shown. These chords are major, minor and diminished chords, using various inversions (1st, 2nd).

C	F A C	= F
	E A C	= Am 2nd
	E G C	= C 1st
B	E G B	= Em
	D G B	= G 2nd
	D F B	= B dim 1st
A	D F A	= Dm
	C F A	= F 2nd
	C E A	= Am 1st
G	C E G	= C
	B E G	= Em 2nd
	B D G	= G 1st
F	B D F	= B dim
	A D F	= Dm 2nd
	A C F	= F 1st
E	A C E	= Am
	G C E	= C 2nd
	G B E	= Em 1st
D	G B D	= G
	F B D	= B dim 2nd
	F A D	= Dm 1st

Triads for C maj. key

The left table shows all the possible $21 = 7 \cdot 3$ triads for the seven tones of the C maj. key (C, B, A, G, F, E, D). Every "key-inherent triad" occurs three times, in its three inversions: root position, 1st inversion and 2nd inversion.

→ 7th chords (= special 4-note chords) - SEPTAKKORDE - result by stacking minor and major 3rds. The "C-major-key-inherent" 7th chords are shown in the right figure.



• More on Chords...

8/29/2024

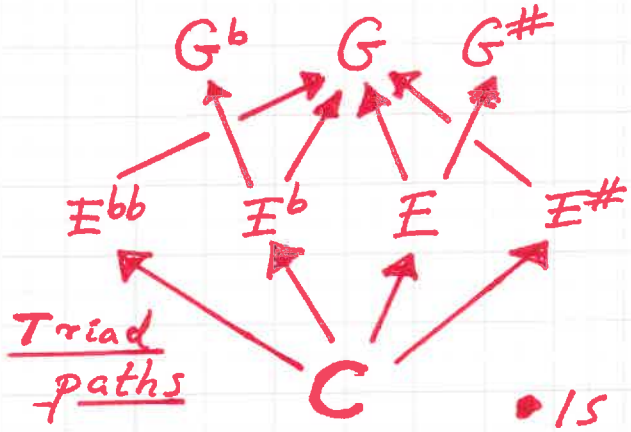


• Common triads (Cmaj key):
CE^bG^b, $CE^{bb}G (= CDG)$, CEG,
CE^bG, $CE^\#G (= CFG)$, $CEG^\#$

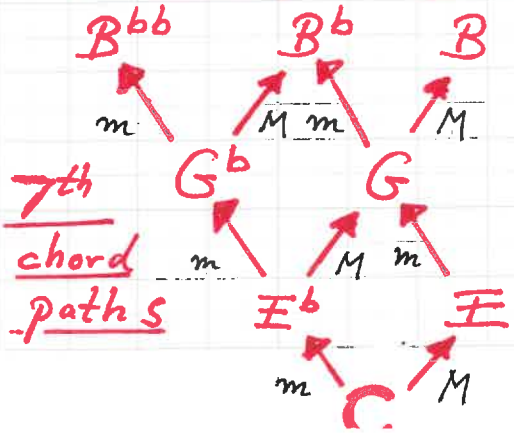
• Notation (left figure):
dim = diminished, aug = augmented,
s2 = sus2, s4 = sus4

• ISSUE: Can, for example, dim, s2 and s4 be included in an "expanded 7th chord set"?

• Can use a graph/network layout to visualize chords. Possible/allowable/used chords can be understood as (directed) paths with 3 nodes (=notes) connected by 2 edges, see left figure.



• IS IT POSSIBLE TO DEFINE A "PHYLOGENETIC DISTANCE" FOR CHORDS (similar to genetic distance for species' genomes) USING SUCH DISCRETE GRAPH/NETWORK (TREE? PATH?) CONCEPTS — to identify smooth progressions, resolutions, modulations etc.?

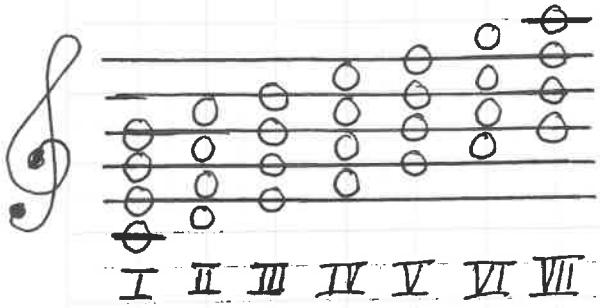


• The left figure shows the six "used" 7th chords in the key of Cmaj as stacks of minor (m) and major (M) 3rds. The six chords correspond to the six "up-paths."

• More on Chords — Focus on 7th Chords

8/29/2024

Example: "Cmaj-inherent" 7th chords



• Notation: m = minor 3rd
M = major 3rd

I	C E G B	M m M
II	D F A C	m M m
III	E G B D	m M m
IV	F A C E	M m M
V	G B D F	M m m
VI	A C E G	m M m
VII	B D F A	m m M

⇒ Each of the seven notes C, D, E, F, G, A and B is in four of these seven "Cmaj-inherent" 7th chords. Thus, one can construct four 7th chords for each of the seven notes, when the note is used as the "melody note" at the top.

B	C E G B	I⁰
	D E G B	III¹
	D F G B	V²
	D F A B	VII³
C	D F A C	II⁰
	E F A C	IV¹
	E G A C	VI²
	E G B C	I³
D	E G B D	III⁰
	F G B D	V¹
	F A B D	VII²
	F A C D	II³
E	F A C E	IV⁰
	G A C E	VI¹
	G B C E	I²
	G B D E	III³
F	G B D F	V⁰
	A B D F	VII¹
	A C D F	II²
	A C E F	IV³
G	A C E G	VI⁰
	B C E G	I¹
	B D E G	III²
	B D F G	V³
A	B D F A	VII⁰
	C D F A	II¹
	C E F A	IV²
	C E G A	VI³

• Notation: $I^0 \hat{=}$ chord I, root position
$III^1 \hat{=}$ chord III, 1st inversion
$V^2 \hat{=}$ chord V, 2nd inversion
$VII^3 \hat{=}$ chord VII, 3rd inversion
⇒ same for all other chords!

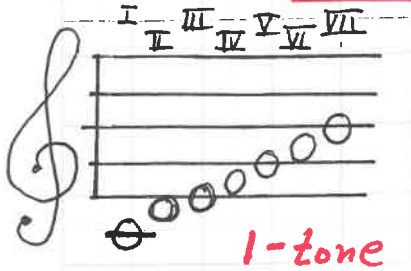
→ The left table shows all possible 4·7 = 28 inversions possible for the "C-maj-inherent" 7th chords — using each Cmaj-note four times as the "top note."
(The same kind of 28 inversions can be compiled for the "C-min-inherent" 7th chords, keeping in mind that chord V has a G♯ in it — not a G.)

→ In classical music theory, all these chords are precisely named (REF: RAMEAU).

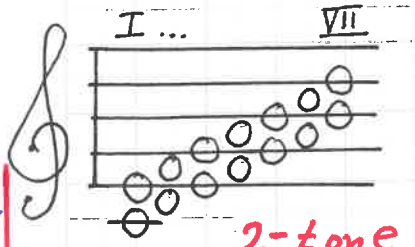
• More on Chords... 8/30/2024

"Summary of 1-tone, 2-tone (3rds), 3-tone (triads) and 4-tone (7ths) chords — and their inversions"

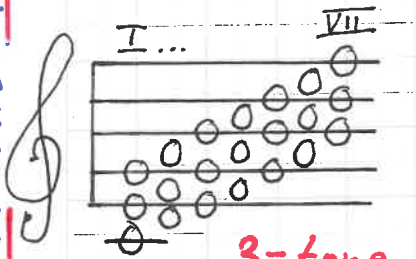
WHEN A CHORD IS RECOGNIZED IN A PIECE, THE PEDAL CAN/SHOULD BE/MUST BE PRESSED FOR THE DURATION OF THE CHORD IN A PIECE PASSAGE.



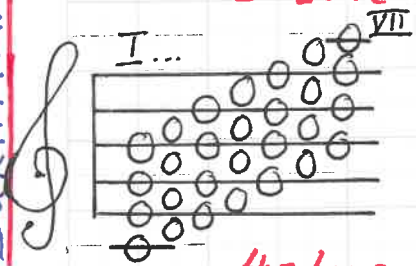
1-tone



2-tone



3-tone



4-tone

• 1-tone/single note: C, D, E, F, G, A, B (= single-note "chords" in Cmaj.):
⇒ only one "inversion" exists: $C^{o}, D^{o}, E^{o}, F^{o}, G^{o}, A^{o}, B^{o}$
(superscript "o" denoting root position)

• 2-note chords (3rds only) in key of Cmaj:
Every note of the Cmaj scale can be played as "melody-note" by using a "key-inherent 3rd" or its (first and only) inversion (3rd or 6th chord), see right figure.

C	I°	C
D	II°	D
E	III°	E
F	IV°	F
G	V°	G
A	VI°	A
B	VII°	B

A C	VI°	C
E C	I¹	
B D	VII°	D
F D	II¹	
C E	I°	E
G E	III¹	
D F	II°	F
A F	IV¹	
E G	III°	G
B G	V¹	
F A	IV°	A
C A	VI¹	
G B	V°	B
D B	VII¹	

• 3-note chords (triads, stacks of two 3rds only) in key of Cmaj.: Every "key-inherent" "melody note" can be represented/played in three ways, using a root position, 1st or 2nd inversion chord ((3rd+3rd)-chord, (3rd+4th)-chord, (4th+3rd)-chord).

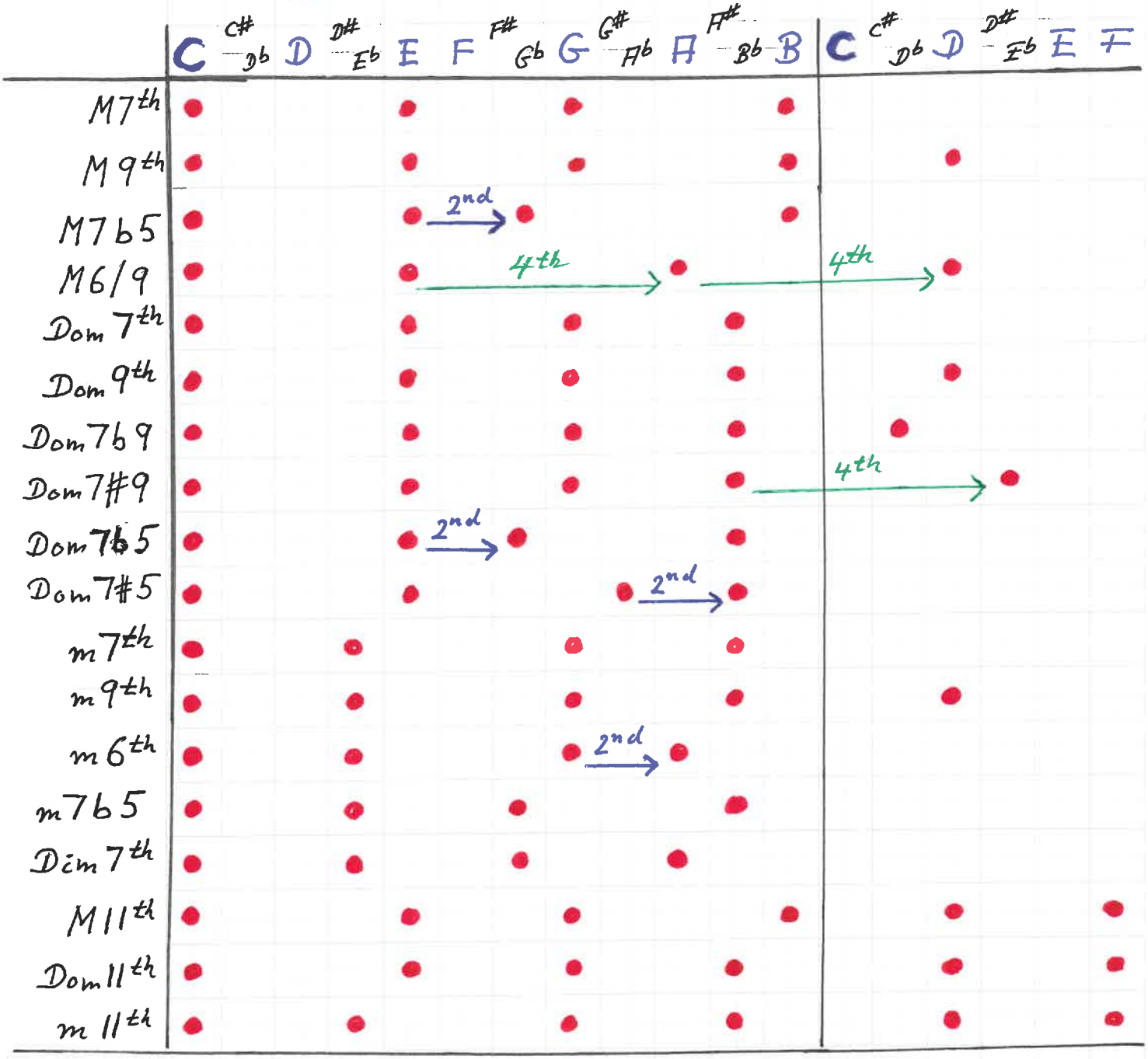
F A C	IV°	C
E G C	I¹	
E A C	VI²₄	

D F A C	II°	C
E G B C	I¹	
E G A C	VI²₄	
E F A C	IV³	

• 4-note chords (7th chords only): Every "key-inherent" "melody note" can be represented/played in four ways, using a root position, 1st, 2nd or 3rd inversion chord ((3rd+3rd+3rd)-chord, (3rd+3rd+2nd)-chord, (3rd+2nd+3rd)-chord, (2nd+3rd+3rd)-chord).

"JAZZ CHORDS"

9/1/2024



M = major

m = minor

Dom = dominant

Dim = diminished

"Construction":

$N1 \xrightarrow{I_1} \ldots \xrightarrow{I_5} N6$

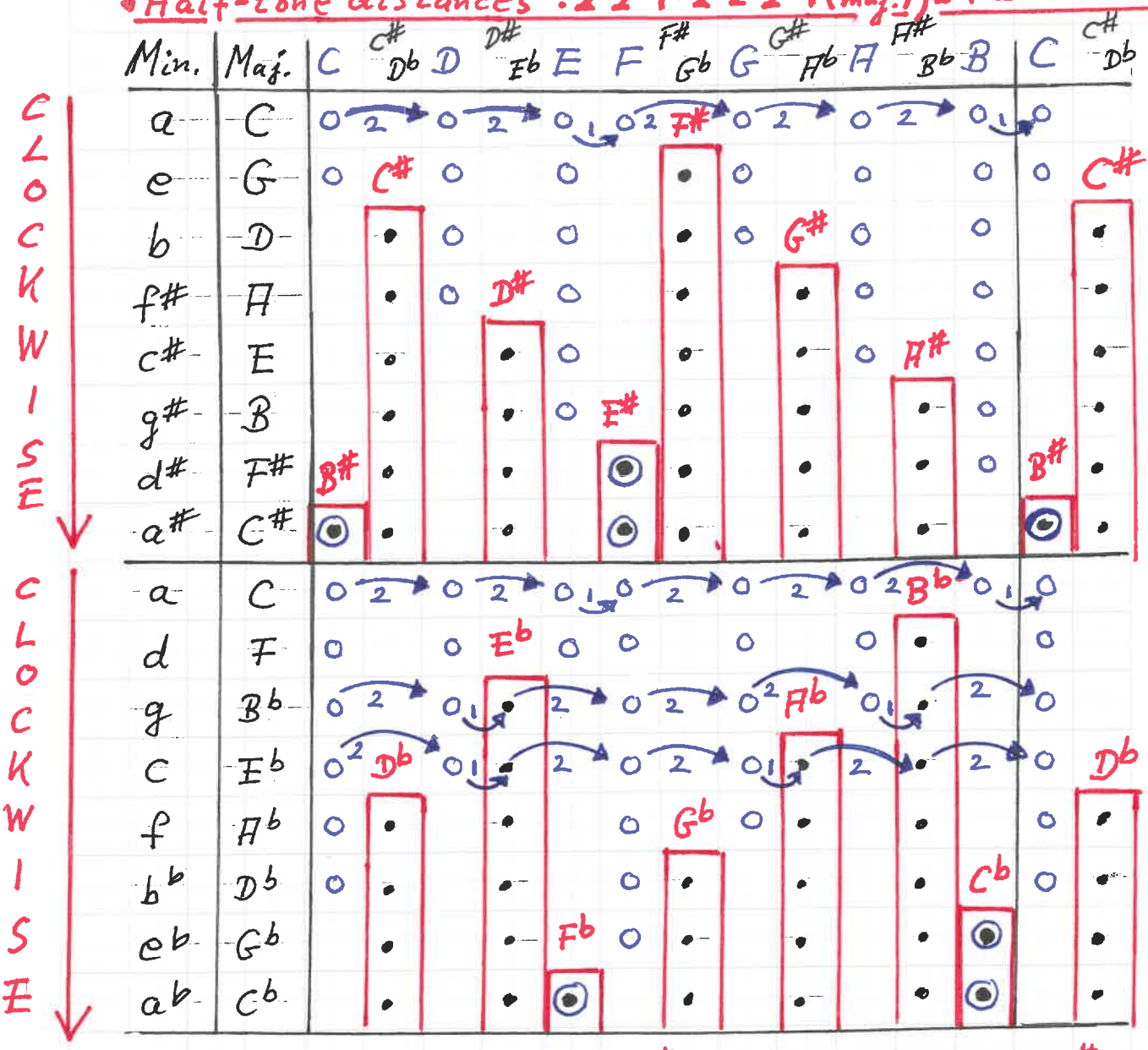
N = note, $I_j \in \{m3^{rd}, M3^{rd}, 2^{nd}, 4^{th}\}$

Most of these "Jazz Chords" (shown only in ROOT position in the key of C) are stacks of minor and major 3rds; only few chords contain a 2nd or 4ths. Often, inversions contain "dense note clusters."

On the Circle of Fifths — and the "Appearance" of '#' and 'b'

9/2/2024

- Half-tone "distances": 2-2-1-2-2-2-1 (maj.), 2-1-2-2-1-2-2 (min.)



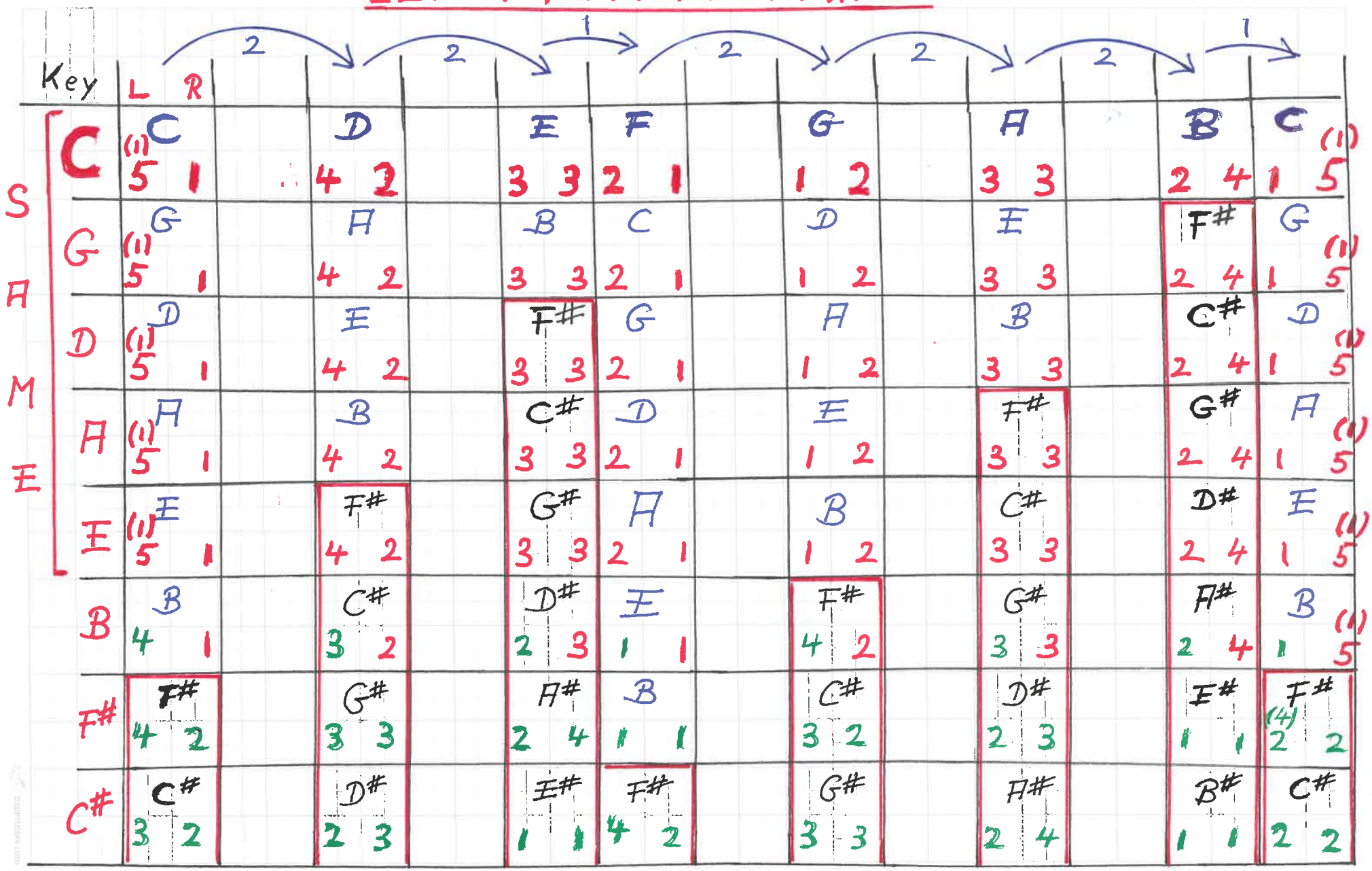
- Maj. scales – "clockwise": C, G, D, A, E, B, F#, C#
  – "counter-clockwise": C, F, Bb, Eb, Ab, Db, Gb, Cb
- Min. scales – "clockwise": a, e, b, f#, c#, g#, d#, a#
  – "counter-clockwise": a, d, g, c, f, bb, eb, ab

- Order of introduction of '#': F#, C#, G#, D#, A#, E# (≙F), B# (≙C)
- Order of introduction of 'b': Bb, Eb, Ab, Db, Gb, Cb (≙B), Fb (≙E)

• FINGERING — MAJOR SCALES (#),

9/3/2024

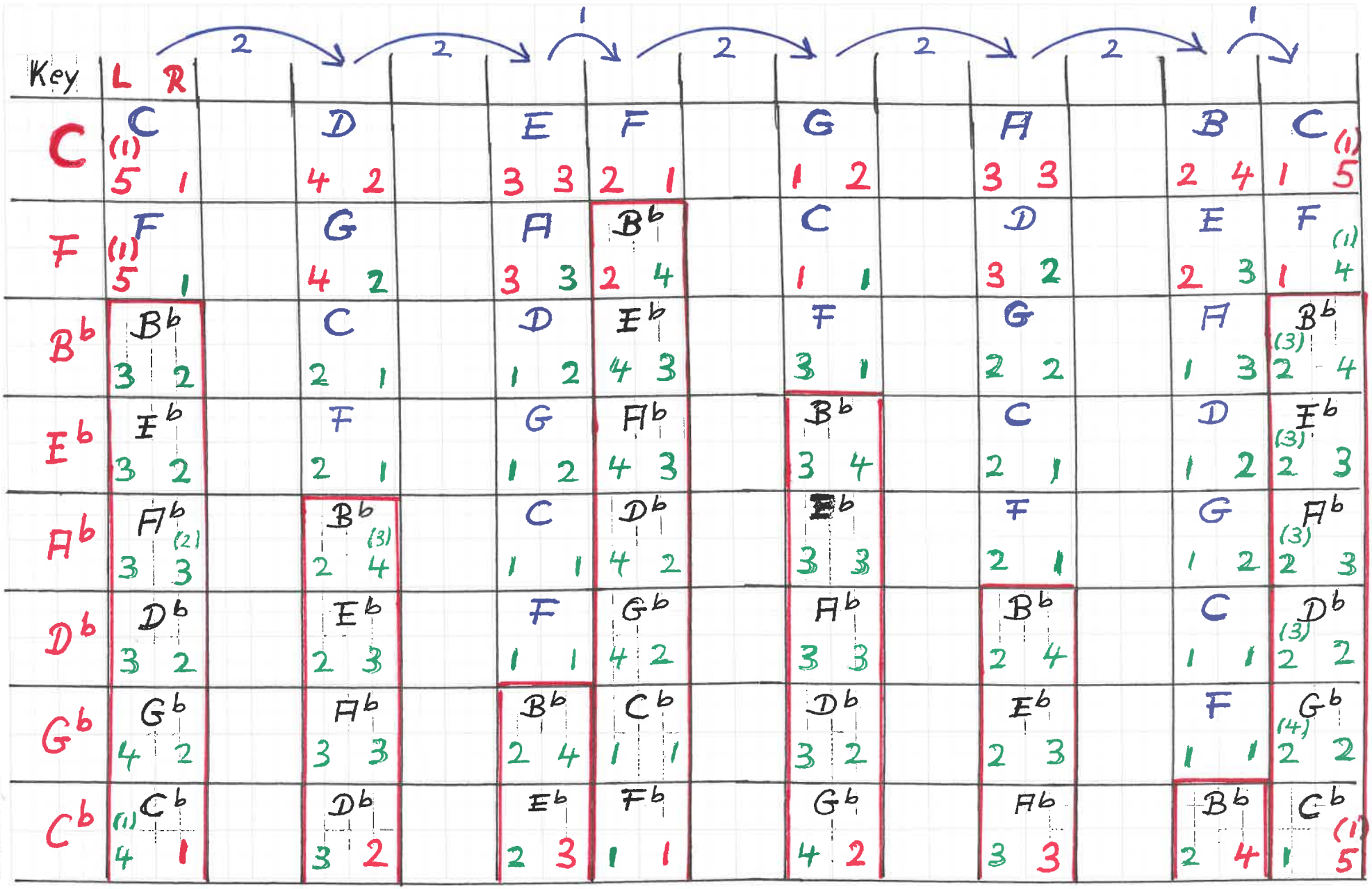
LEFT (L) & RIGHT (R) HAND



(..)(..) — when continuing (another octave)

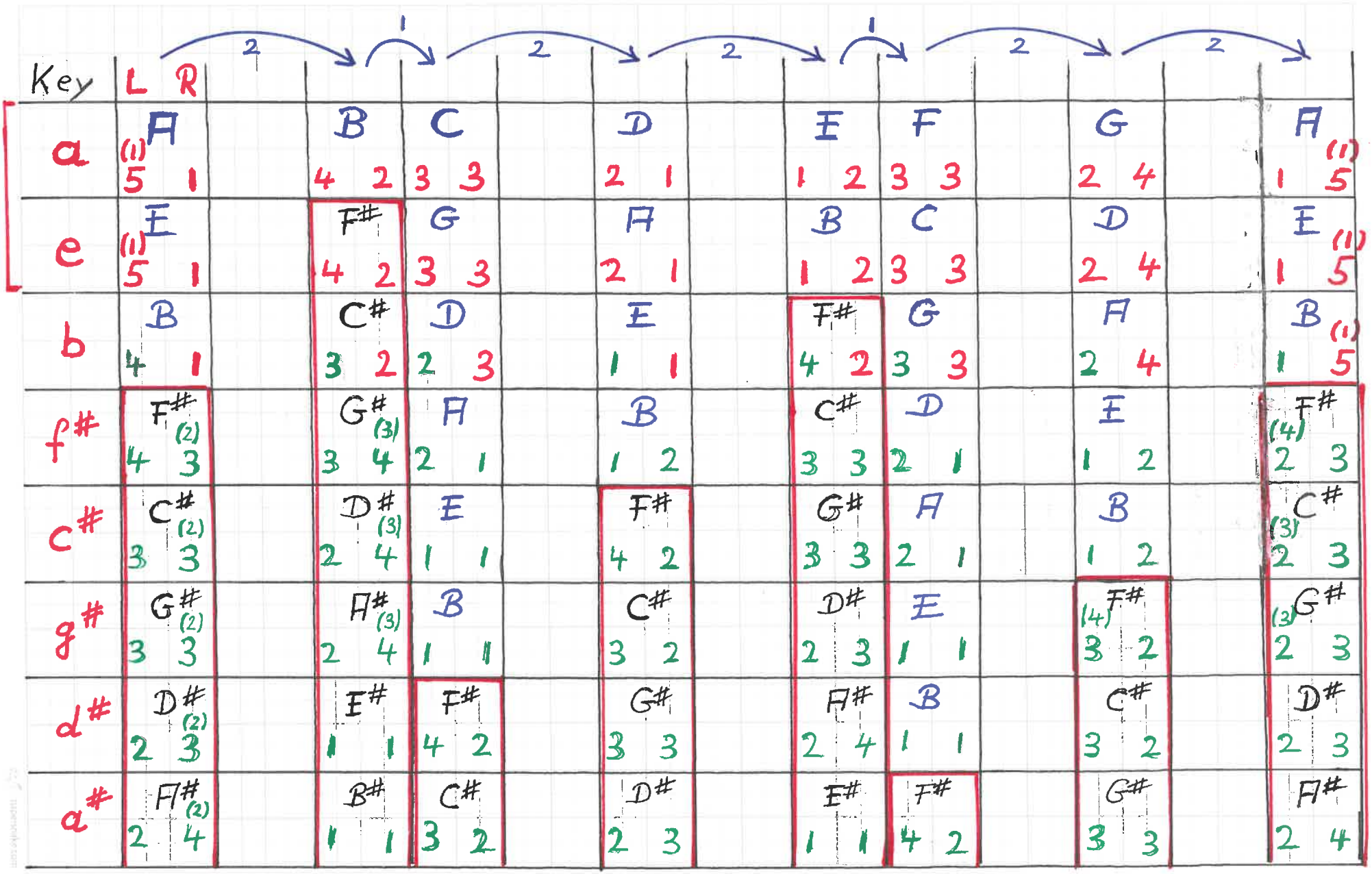
FINGERING — MAJOR SCALES (b), LEFT (L) & RIGHT (R) HAND

9/4/2024



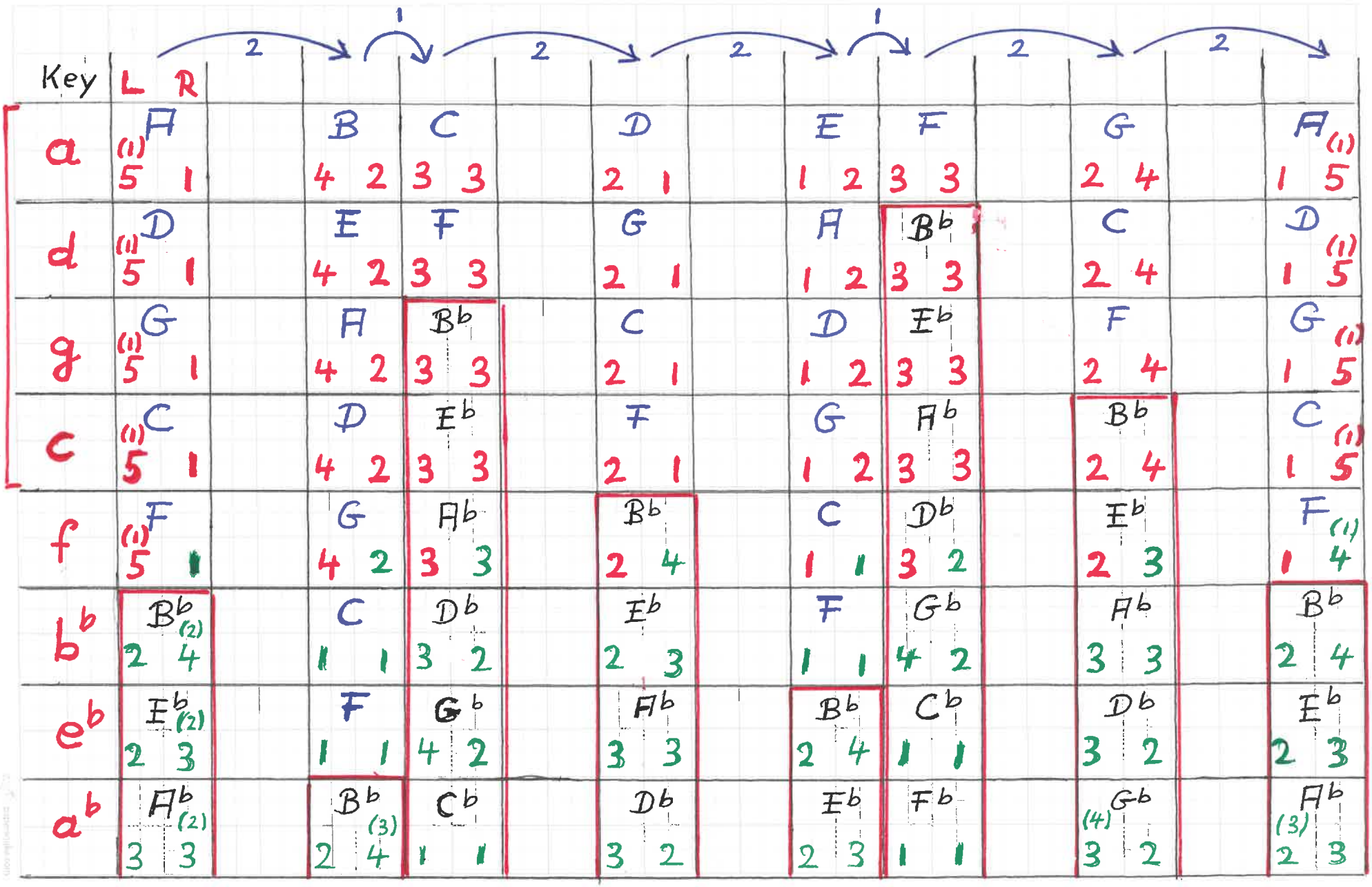
• FINGERING — MINOR SCALES (#), LEFT (L) & RIGHT (R) HAND

9/5/2024



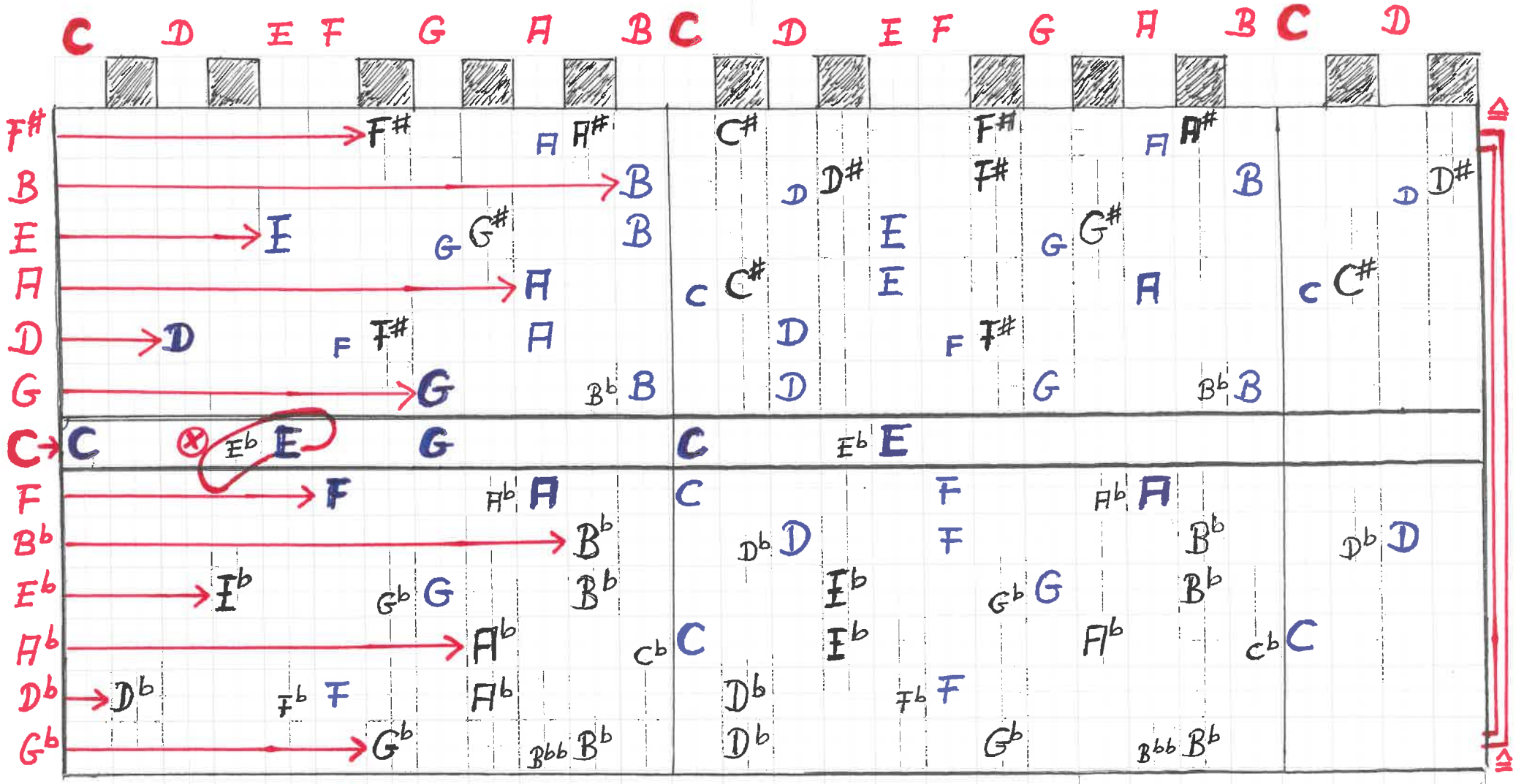
FINGERING — MINOR SCALES (b), LEFT (L) & RIGHT (R) HAND

9/5/2024



• MAJOR & MINOR TRIADS WITH INVERSIONS ⊛

9/6/2024



	L	R
0th	5, 3, 1	1, 3, 5
1st	5, 3, 1	1, 2, 5
2nd	5, 2, 1	1, 3, 5

FINGERING

The left table shows the standard fingerings for all major and minor triads and all inversions.

⊗ E♭ E "NOTE PAIR" for minor & major keys

An "octave chord," e.g. C-E-G-C, uses the standard fingerings 5-4-2-1 (L) and 1-2-3-5 (R).

⊛ Here: C-E-G-C-E (0th, 1st, 2nd inversions)

9/7/2024

• DIATONIC TRIADS — MINOR KEYS

(Key-inherent triads and their "implied inversions")

Key	i	ii°	III	iv	v	VI	VII
a#	A#C#E#	B#D#F#	C#E#G#	D#F#A#	E#G#B#	F#A#C#	G#B#D#
d#	D#F#A#	E#G#B	F#A#C#	G#B D#	A#C#E#	B D#F#	C#E#G#
g#	G#B D#	A#C#E	B D#F#	C#EG#	D#F#A#	EG#B	F#A#C#
c#	C#EG#	D#F#A	EG#B	F#AC#	G#B D#	AC#E	B D#F#
f#	F#AC#	G#B D	AC#E	B DF#	C#EG#	DF#A	EG#B
b	B DF#	C#EG	DF#A	EGB	F#AC#	GB D	AC#E
e	EGB	F#AC	GB D	ACE	B DF#	CEG	DF#A
a	ACE	B DF	CEG	DFA	EGB	FAC	GB D
d	DFA	EGB♭	FAC	GB♭D	ACE	B♭DF	CEG
g	GB♭D	ACE♭	B♭DF	CE♭G	DFA	E♭GB♭	FAC
c	CE♭G	DFA♭	E♭GB♭	FA♭C	GB♭D	A♭CE♭	B♭DF
f	FA♭C	GB♭D♭	A♭CE♭	B♭D♭F	CE♭G	D♭FA♭	E♭GB♭
b♭	B♭D♭F	CE♭G♭	D♭FA♭	E♭G♭B♭	FA♭C	G♭B♭D♭	A♭CE♭
e♭	E♭G♭B♭	FA♭C♭	G♭B♭D♭	A♭C♭E♭	B♭D♭F	C♭E♭G♭	D♭FA♭
a♭	A♭C♭E♭	B♭D♭F♭	C♭E♭G♭	D♭F♭A♭	E♭G♭B♭	F♭A♭C♭	G♭B♭D♭

EQUIVALENT!

\# : i, ii°, III, iv, v, VI, VII

♭ : i, ii°, III, iv, v, VI, VII

This table concerns the NATURAL MINOR scales. Regarding the HARMONIC and MELODIC MINOR scales, the 6th and 7th scale degrees (notes) must be raised by a halftone (#). This affects all triads (ii°–VII), except i.

III, VI, VII — major chords / triads

i, iv, v — minor chords / triads

ii° — diminished (minor) chord / triad

⇒ Triad types: M 3rd m 3rd, m 3rd M 3rd / m 3rd m 3rd !

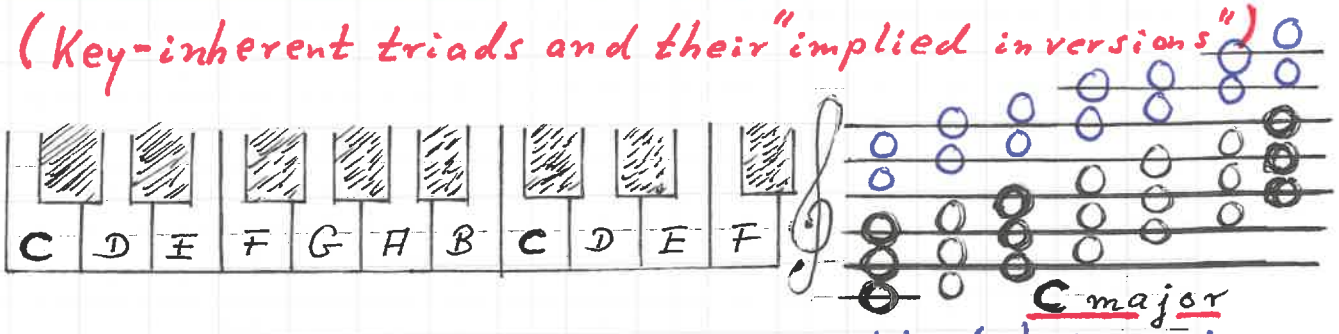
• Note. The keys of the minor keys used for the above table are keys of the NATURAL MINORS. Thus, the seven triads for the key of a-minor, for example, are the same triads shown for C-major — with different numbering.

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9/7/2024

• DIATONIC TRIADS — MAJOR KEYS

(Key-inherent triads and their "implied inversions")



blue 'o': inversions

Key	I	ii	iii	IV	V	vi	vii°
C#	C#E#G#	D#F#A#	E#G#B#	F#A#C#	G#B#D#	A#C#E#	B#D#F#
F#	F#A#C#	G#B#D#	A#C#E#	B#D#F#	C#E#G#	D#F#A#	E#G#B#
B	BD#F#	C#EG#	D#F#A#	EG#B	F#A#C#	G#BD#	A#C#E
E	EG#B	F#AC#	G#BD#	AC#E	BD#F#	C#EG#	D#F#A
A	AC#E	BDF#	C#EG#	DF#A	EG#B	F#AC#	G#BD
D	DF#A	EGB	F#AC#	GBD	AC#E	BDF#	C#EG
G	GBD	ACE	BDF#	CEG	DF#A	EGB	F#AC
C	CEG	DFA	EGB	FAC	GBD	ACE	BDF
F	FAC	GBbD	ACE	BbDF	CEG	DFA	EGBb
Bb	BbDF	CEbG	DFA	EbGBb	FAC	GBbD	ACEb
Eb	EbGBb	FAbC	GBbD	AbCEb	BbDF	CEbG	DFAb
Ab	AbCEb	BbDbF	CEbG	DbFAb	EbGBb	FAbC	GBbDb
Db	DbFAb	EbGbBb	FAbC	GbBbDb	AbCEb	BbDbF	CEbGb
Gb	GbBbDb	AbCbEb	BbDbF	CbEbGb	DbFAb	EbGbBb	FAbCb
Cb	CbEbGb	DbFbAb	EbGbBb	FbAbCb	GbBbDb	AbCbEb	BbDbFb

= EQUIVALENT!

I, IV, V — major chords / triads

ii, iii, vi — minor chords / triads

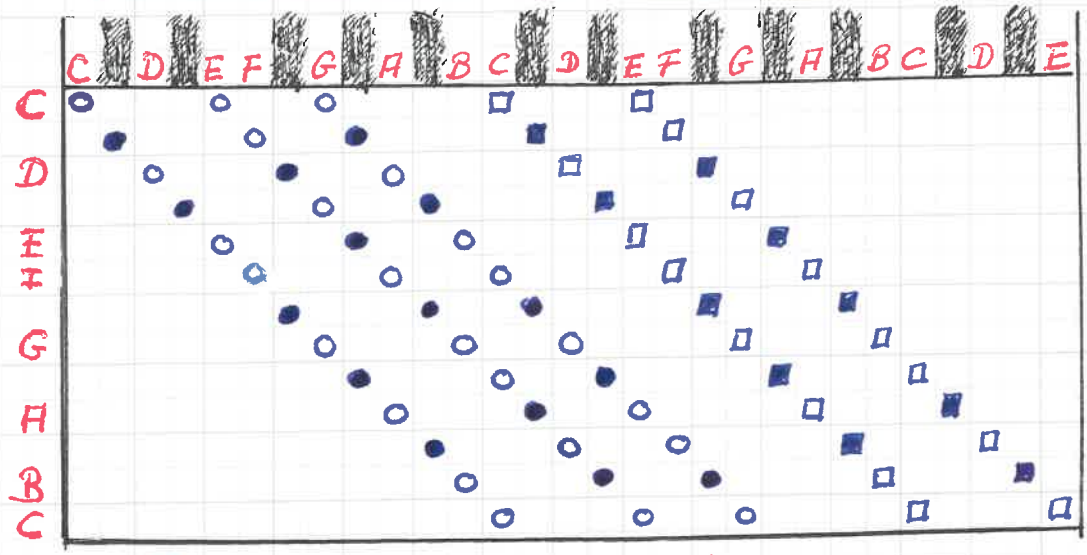
vii° — diminished (minor) chord / triad

⇒ Triad types: M3rd m3rd, m3rd M3rd, m3rd m3rd !

⇒ Three "shapes" only!

9/8/2024

• FINGERING PATTERNS/SHAPES FOR TRIADS & INVERSIONS



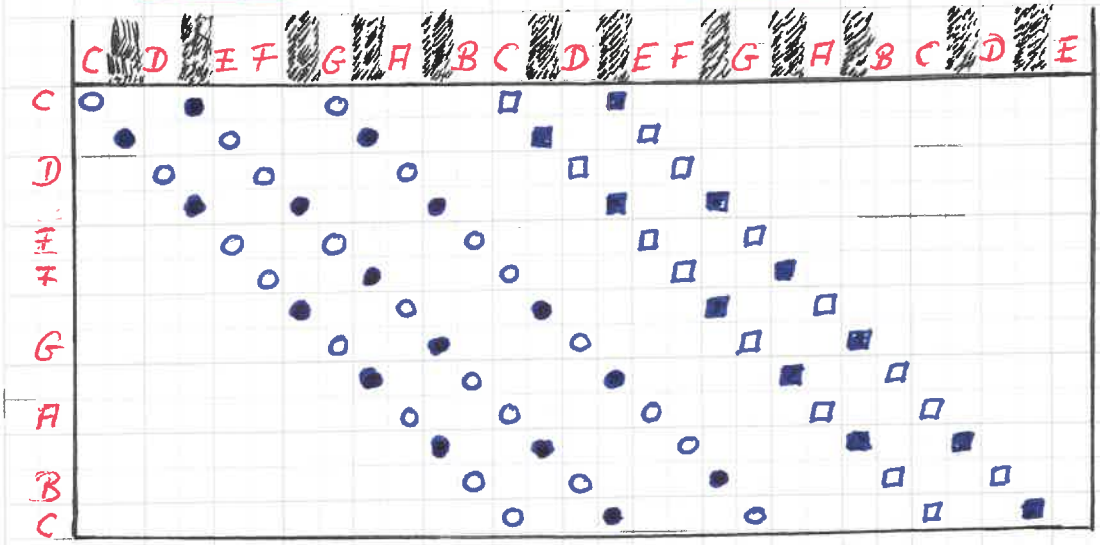
WWW - ww 1
BWB - BW 2
WBW - WB 3
BWB - BW 2
WBW - WB 3
WWW - ww 1
BBB - BB 4
WWW - ww 1
BWB - BW 2
WBW - WB 3
BWW - BW 5
WBB - WB 6
WWW - ww 1

Major Triads = M 3rd m 3rd

o, □ white keys (W) ; •, ■ black keys (B)

WWW white-white-white key pattern etc.

1, ..., 6 six types of patterns for major triads

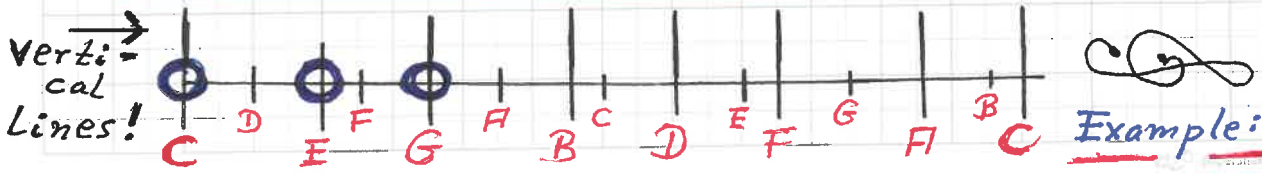


WBW - WB 3
BWB - BW 2
WWW - WW 1
BBB - BB 4
WWW - ww 1
WBW - WB 3
BWB - BW 2
WBW - WB 3
BWB - BW 2
WWW - ww 1
BBW - BB 7
WWB - ww 8
WBW - WB 3

Minor Triads = m 3rd M 3rd

7, 8 two additional types of patterns for minor triads

• "ROTATING SHEET NOTATION BY -90°":



→ Chords ①, ② and ③ (and their inversions) are most common.

→ ALL 16 (=2^4) possible patterns occur in the table, from (○,○,○,○) to (●,●,●,●).

PATTERNS/SHAPES OF 7THS

Key	① M3 m3 M3	② M3 m3 m3	m3 M3 M3	③ m3 M3 m3	m3 m3 M3	m3 m3 m3
C	○ ○ ○ ○	○ ○ ○ ●	○ ● ○ ○	○ ● ○ ●	○ ● ● ●	○ ● ● ○
	● ○ ● ○	● ○ ● ●	● ○ ● ○	● ○ ● ○	● ○ ○ ○	● ○ ○ ●
D	○ ● ○ ●	○ ● ○ ○	○ ○ ○ ●	○ ○ ○ ○	○ ○ ● ○	○ ○ ● ○
	● ○ ● ○	● ○ ● ●	● ● ● ○	● ● ● ●	● ● ○ ○	● ● ○ ○
E	○ ● ○ ●	○ ● ○ ○	○ ○ ○ ●	○ ○ ○ ○	○ ○ ● ○	○ ○ ● ●
F	○ ○ ○ ○	○ ○ ○ ●	○ ● ○ ○	○ ● ○ ●	○ ● ○ ○	○ ● ○ ●
	● ● ● ○	● ● ● ○	● ○ ● ○	● ○ ● ○	● ○ ○ ○	● ○ ○ ●
G	○ ○ ○ ●	○ ○ ○ ○	○ ● ○ ●	○ ● ○ ○	○ ● ● ○	○ ● ● ○
	● ○ ● ○	● ○ ● ●	● ○ ● ○	● ○ ● ●	● ○ ○ ●	● ○ ○ ○
A	○ ● ○ ●	○ ● ○ ○	○ ○ ○ ●	○ ○ ○ ○	○ ○ ● ○	○ ○ ● ●
	● ○ ○ ○	● ○ ○ ●	● ● ○ ○	● ● ● ●	● ● ○ ○	● ● ○ ○
B	○ ● ○ ●	○ ● ○ ○	○ ○ ● ●	○ ○ ● ○	○ ○ ○ ○	○ ○ ○ ●

→ Notation: M3 m3 M3 ≙ root + M3rd + m3rd + M3rd etc. (M=major, m=minor)

→ See handwritten notes from 8/29/2024 for detail for a 7th chord.

→ Fingering a 7th chord: L: 5-3-2-1 (or 5-4-2-1), R: 1-2-3-5 (or 1-2-4-5)

→ Use for accompaniment of a melody: Combination of "key-inherent" triads and 7th chords usable to accompany melody in a specific key.
Ex.: Key = C major ⇒ allowable chords: F-A-C-E, G-B-D, C-E-G, A-C-E, ...

9/9/2024

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PATTERNS/SHAPES OF CHORDS — VIEWED AS WHITE-BLACK SEQUENCES

9/9/2024

TRIADS

Key	Major	Rel. Minor
C	○ ○ ○	○ ● ○
	● ○ ●	● ○ ●
D	○ ● ○	○ ○ ○
	● ○ ●	● ● ●
E	○ ● ○	○ ○ ○
F	○ ○ ○	○ ● ○
	● ● ●	● ○ ●
G	○ ○ ○	○ ● ○
	● ○ ●	● ○ ●
A	○ ● ○	○ ○ ○
	● ○ ○	● ● ○
B	○ ● ●	○ ○ ●
C	○ ○ ○	○ ● ○

→ The left table summarizes the possible triad key patterns as sequences of white (○) and black (●) keys.

→ A major triad is understood as a triple (root, M3rd, m3rd), and a relative minor triad is understood as a triple (root, m3rd, M3rd).

→ The actual "physical spacing" between the keys is transparent.

→ All eight ($=2^3$) pattern possibilities occur, from (○, ○, ○) to (●, ●, ●).

Key	THIRDS Major	THIRDS Minor	FIFTHS		FOURTHS	
C	○ ○	○ ●	○ ○		○ ○	
	● ○	● ○	● ●		● ●	
D	○ ●	○ ○	○ ○		○ ○	
	● ○	● ●	● ●		● ●	
E	○ ●	○ ○	○ ○		○ ○	
F	○ ○	○ ●	○ ○		○ ●	x
	● ●	● ○	● ●		● ○	x
G	○ ○	○ ●	○ ○		○ ○	
	● ○	● ○	● ●		● ●	
A	○ ●	○ ○	○ ○		○ ○	
	● ○	● ●	● ○	x	● ●	
B	○ ●	○ ○	○ ●	x	○ ○	
C	○ ○	○ ●	○ ○		○ ○	

THIRDS, FIFTHS & FOURTHS

→ The left table shows key patterns for 2-note chords, i.e., thirds, fifths and fourths.

→ The patterns for (major and minor) thirds and fifths can be extracted from the triads.

→ The patterns for fifts and fourths are always "○○" or "●●" — except for 'x'.

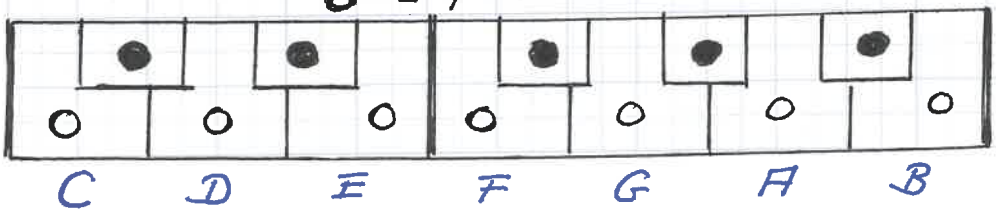
• PATTERNS OF TRIADS — FOR ALL MAJOR & MINOR CHORDS

9/13/2024

Goal:
- → Memorize the SHAPES of all triads.
- → Use BINARY PATTERNS of 'o' (WHITE) and '●' (BLACK) sequences to describe the "physical shapes" of triads on the keyboard.
- → Group / classify triads based on their patterns.

MAJOR	Pattern	minor
C, F, G	o o o (M3 m3 / m3 M3)	d, e, a
D, E, A ✱	●	c, f, g
$D^b(C^\#)$, E^b, A^b	● ●	$c^\#, f^\#, g^\#(a^b)$
$F^\#(G^b)$	● ● ●	$e^b(d^\#)$
B (C^b)	o ● ●	
B^b	● o o	
	o o ●	b
	● ● o	$b^b(a^\#)$

$8 = 2^3$ patterns



- Order of chords could also be based on circle of 5ths.
- The first column shows possible "icons" for the shapes.
- ✱ Example: A ≙ A-$C^\#$-E

→ The Left table uses C, G, D, ... for major chords and c, g, d, ... for minor chords.

→ Memorizing chord shapes eliminates the need to remember the specific notes / note names of chords.

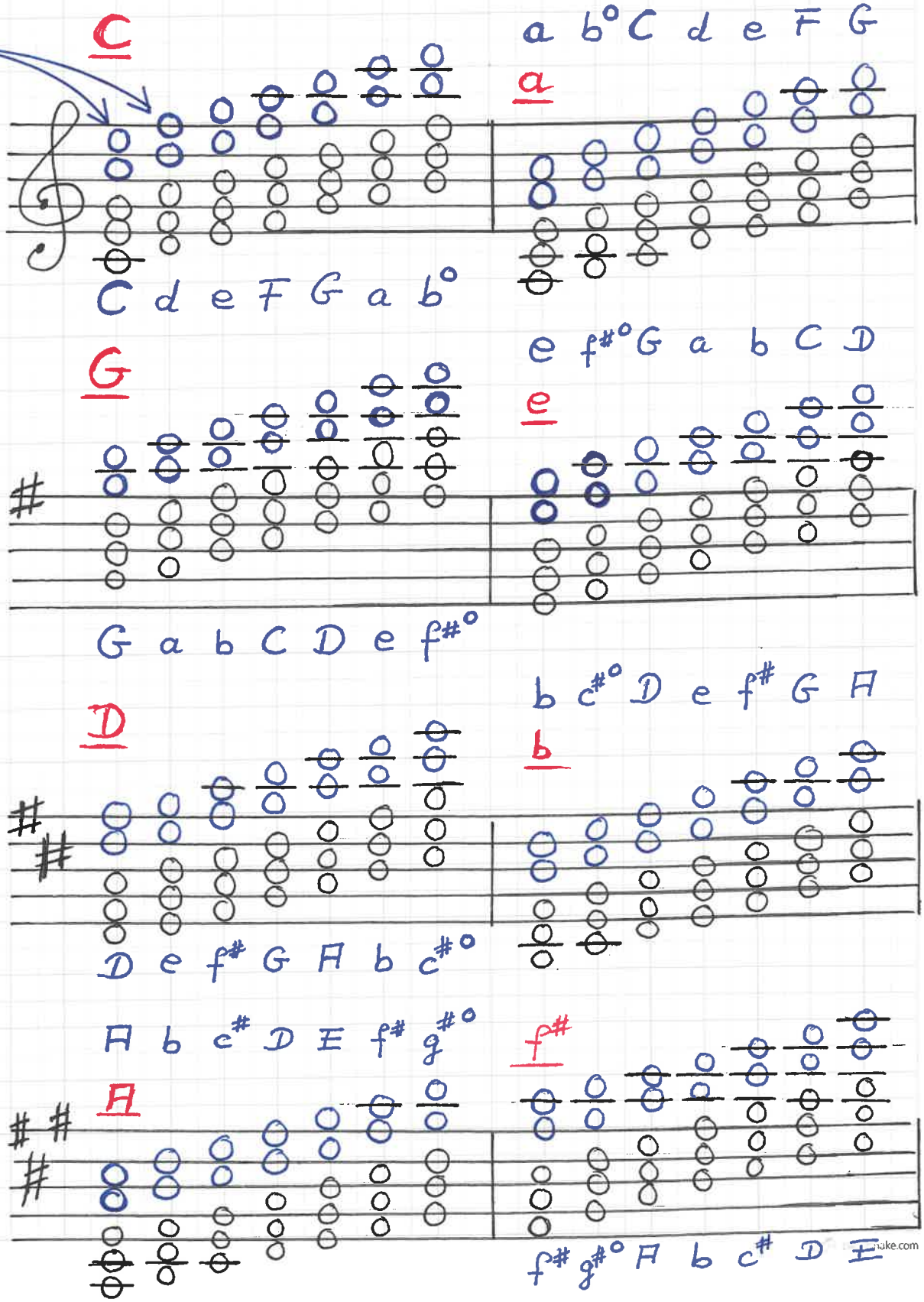
→ These patterns imply the patterns for chord inversions by rearranging the "binary masks" accordingly.

"Key-inherent" Triads for All Keys — Major & Minor Keys and Inversions

9/14/2024

Notation: C, G, ... major keys / chords
a, e, ... minor keys / chords (° dim. chord)

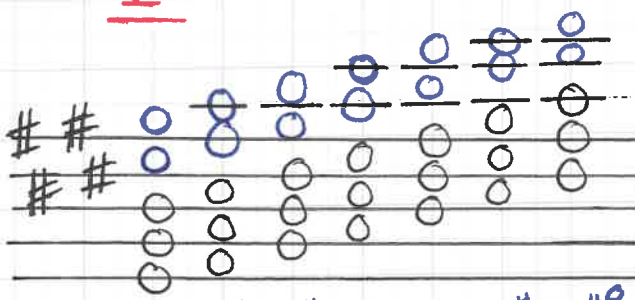
in-versions



"Key-inherent" Triads for All Keys

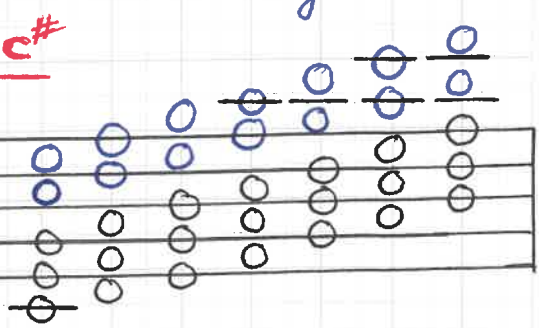
9/14/2024

E



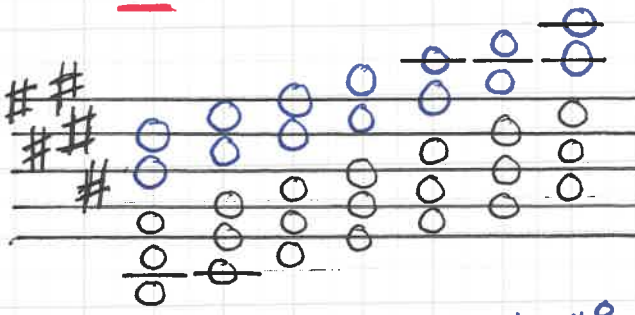
E $f^{\#}$ $g^{\#}$ A B $c^{\#}$ $d^{\#\circ}$

$c^{\#}$



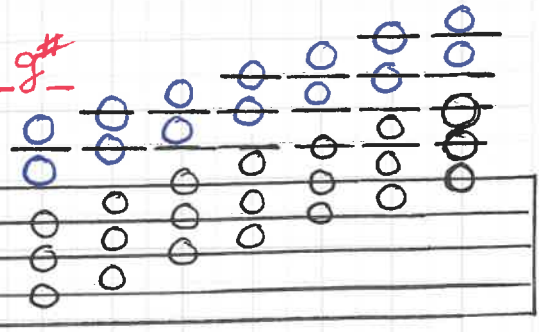
$c^{\#}$ $d^{\#\circ}$ E $f^{\#}$ $g^{\#}$ A B

B



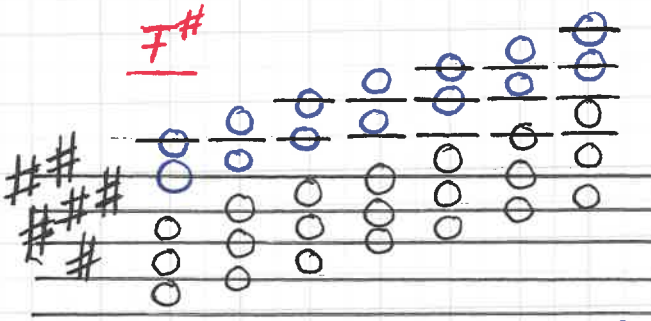
B $c^{\#}$ $d^{\#}$ E $F^{\#}$ $g^{\#}$ $a^{\#\circ}$

$g^{\#}$



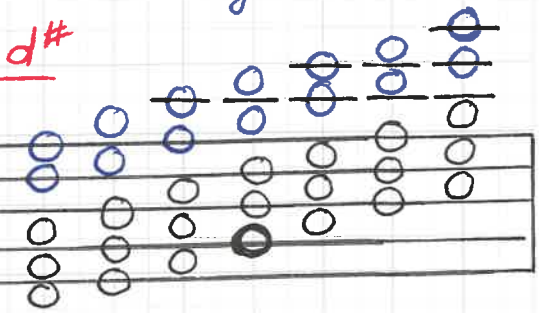
$g^{\#}$ $a^{\#\circ}$ B $c^{\#}$ $d^{\#}$ E $F^{\#}$

$F^{\#}$



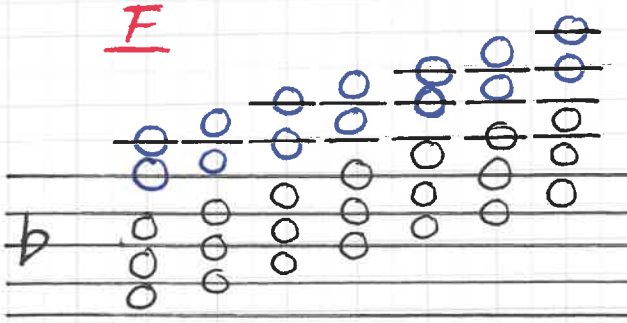
$F^{\#}$ $g^{\#}$ $a^{\#}$ B $C^{\#}$ $d^{\#}$ $e^{\#\circ}$

$d^{\#}$



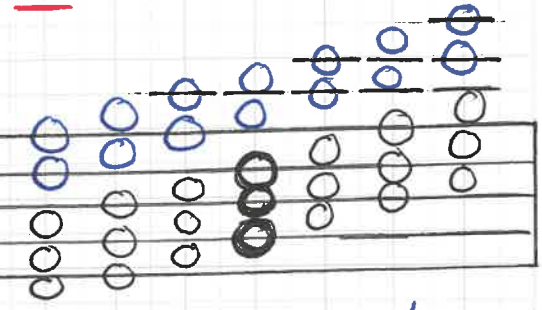
$d^{\#}$ $e^{\#\circ}$ $F^{\#}$ $g^{\#}$ $a^{\#}$ B $C^{\#}$

F



F g a $B^{\flat}$ C d e°

d

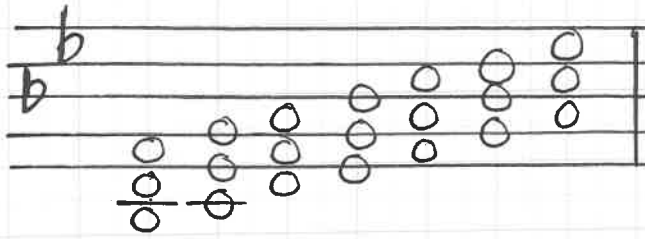


d e° F g a $B^{\flat}$ C

"Key-inherent" Triads for All Keys

9/14/2024

$B^{\flat}$



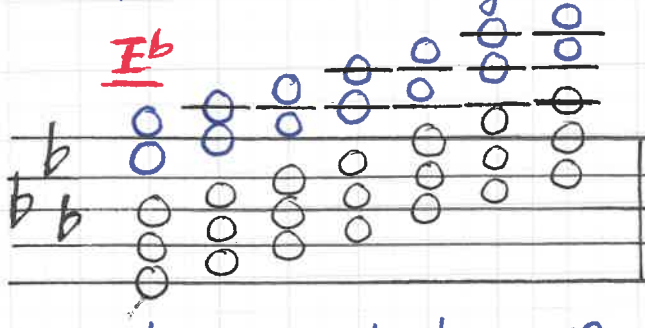
$B^{\flat}$ c d $E^{\flat}$ F g a°

-g-



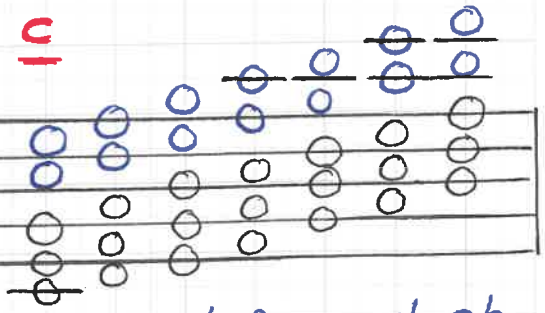
g a° $B^{\flat}$ c d $E^{\flat}$ F

$E^{\flat}$



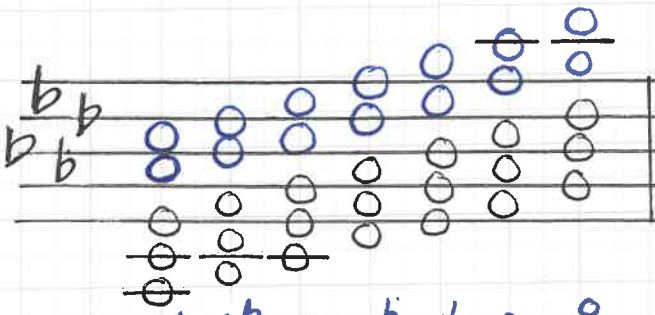
$E^{\flat}$ f g $A^{\flat}$ $B^{\flat}$ c d°

c



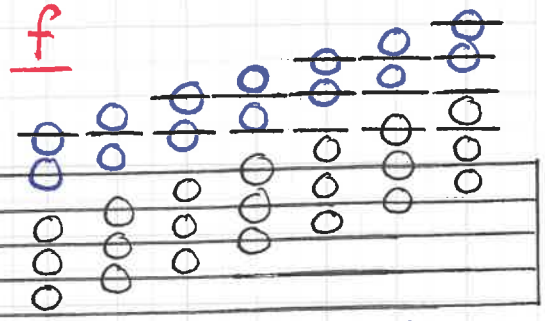
c d° $E^{\flat}$ f g $A^{\flat}$ $B^{\flat}$

$A^{\flat}$



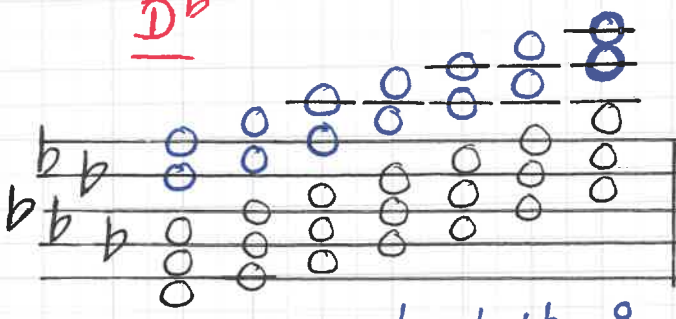
$A^{\flat}$ $b^{\flat}$ c $D^{\flat}$ $E^{\flat}$ f g°

f



f g° $A^{\flat}$ $b^{\flat}$ c $D^{\flat}$ $E^{\flat}$

$D^{\flat}$



$D^{\flat}$ $e^{\flat}$ f $G^{\flat}$ $A^{\flat}$ $b^{\flat}$ c°

$b^{\flat}$

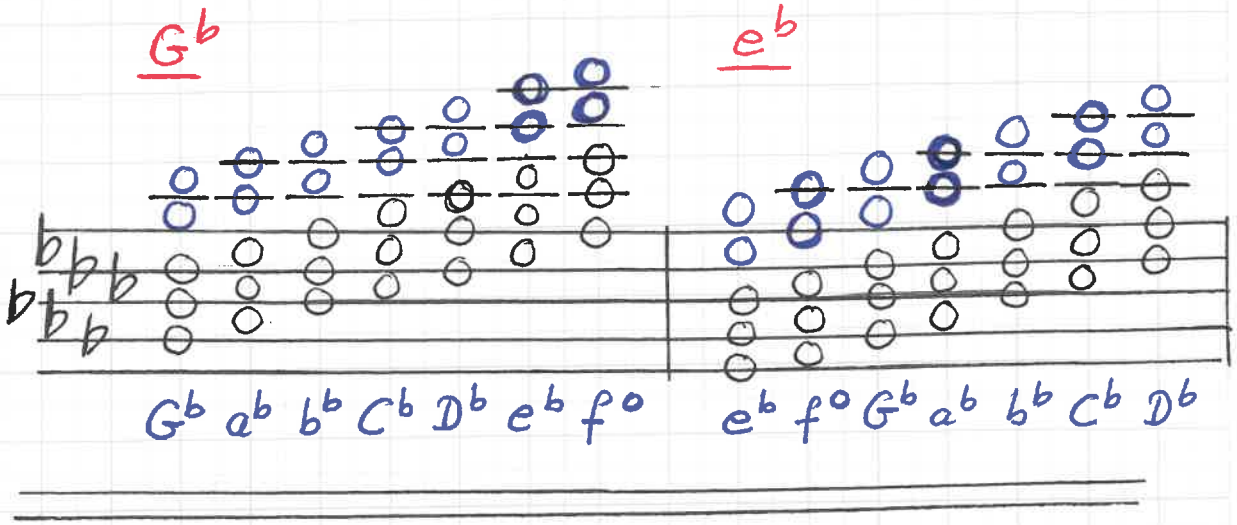


$b^{\flat}$ c° $D^{\flat}$ $e^{\flat}$ f $G^{\flat}$ $A^{\flat}$

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• "Key-inherent" Triads for All Keys

9/14/2024



⇒ **Summary**:

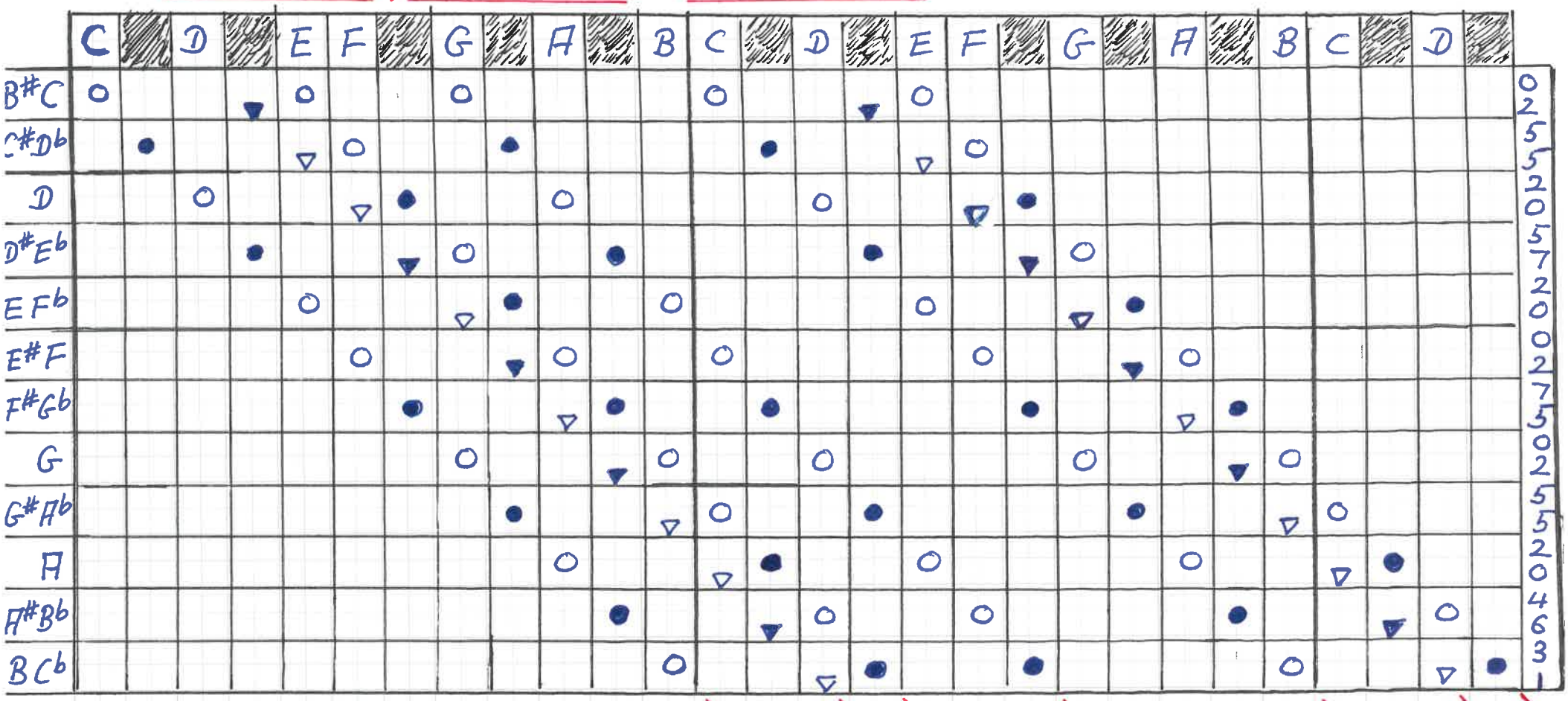
1) One must know/memorize the 7 "key-inherent" chord names of all keys, for major and minor keys. The 7 "key-inherent" chords are 3 major and 3 minor chords and 1 diminished chord. The triads are: C, G, D, A, E, B, $F^\#$; F, B^b, E^b, A^b, D^b, G^b; a, e, b, $f^\#$, $c^\#$, $g^\#$, $d^\#$; and d, g, c, f, b^b, e^b — using large letters for major triads, small letters for minor triads.

2) The 26 triads named in 1) — actually being only 24 triads since $D^b \hat{=} F^\#$ and $e^b \hat{=} d^\#$ — map to the 8 "shapes"/"patterns" for triads (9-13-2024), "white-black key patterns."

3) One must be able to find and play the 8 "patterns" quickly with the right and left hand, using a 1-3-5 and 5-3-1 fingering, respectively.

Triad Patterns for All Major and Minor Keys, Including Inversions

9/15/2024



major & minor triads

for inversions

▽, ▼ for m 3rd of minor chords

0	W	W	W	4	B	W	W
1	W	W	B	5	B	W	B
2	W	B	W	6	B	B	W
3	W	B	B	7	B	B	B

black-white patterns 0...7

→ Triad patterns are Listed in "chromatic order," moving up one half-tone at a time.

→ Each row shows patterns for the major and minor triads, using ▽, ▼ for minor triads.

→ The last column maps triads to patterns.

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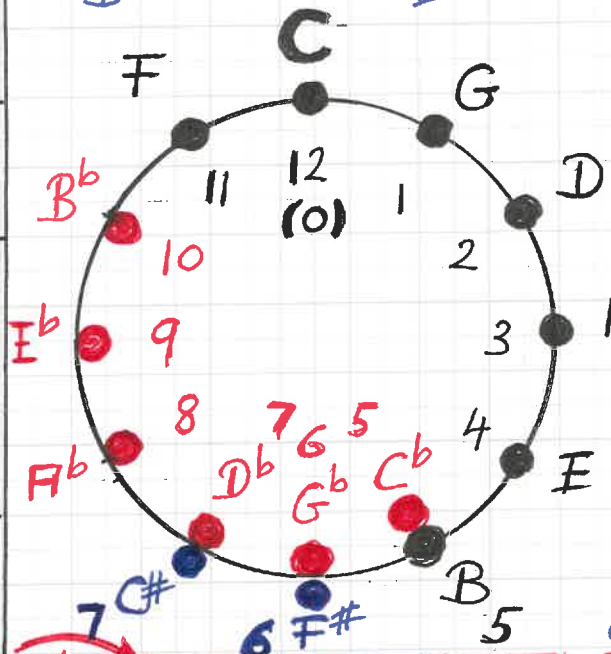
9/16/2024

Major/Minor Triads

MUST MEMORIZE THESE

12 (0)	C	E E^b	G
2	D	$F^\#$ F	A
4	E	$G^\#$ G	B
11	F	A A^b	C
1	G	B B^b	D
3	A	$C^\#$ C	E
5	B	$D^\#$ D	$F^\#$





Principle:

1) The three columns of the three tables all use the same ALPHABETICAL ORDER, i.e., CDEFGAB, EFGABCD, and GABCDEF.

2) The second column lists the M3rd and m3rd notes for major (top) and minor (bottom) triads.

3) The triads with indices 12(0), 1, ..., 11 are "practically" relevant triads.

4) Triad note "LETTERS HAVE ALPHA DISTANCE 2".

(#) 7

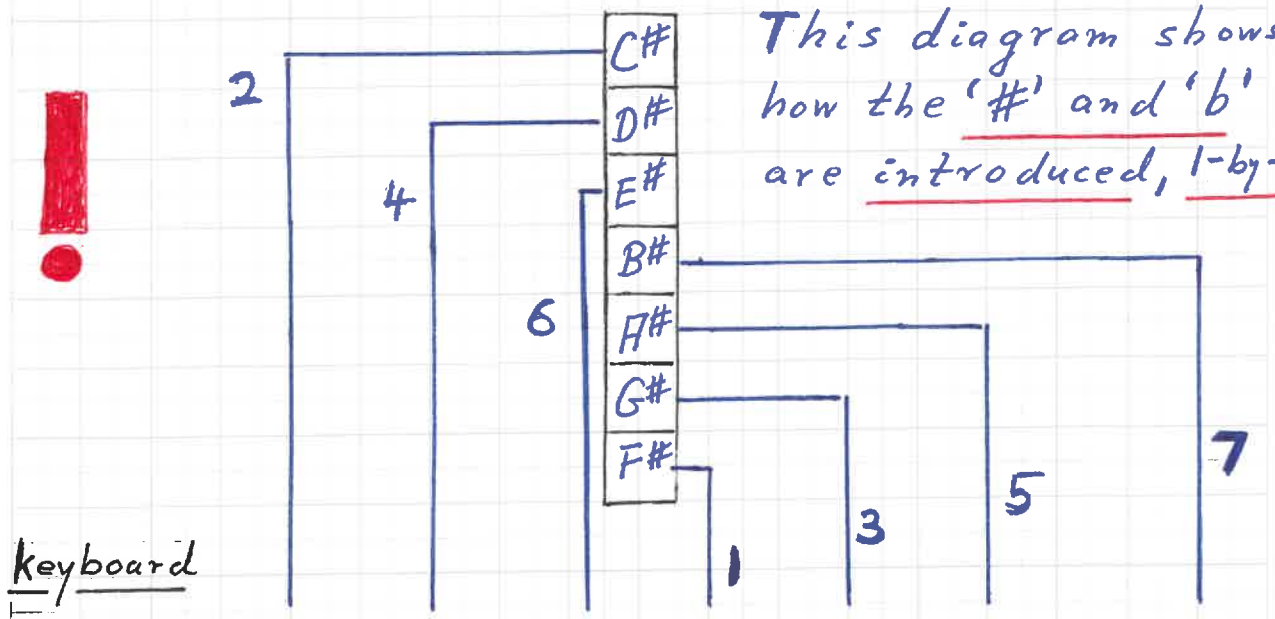
$C^\#$	$E^\#$ E	$G^\#$
$D^\#$	$F^{\#\#}$ $F^\#$	$A^\#$
$E^\#$	$G^{\#\#}$ $G^\#$	$B^\#$
$F^\#$	$A^\#$ A	$C^\#$
$G^\#$	$B^\#$ B	$D^\#$
$A^\#$	$C^{\#\#}$ $C^\#$	$E^\#$
$B^\#$	$D^{\#\#}$ $D^\#$	$F^{\#\#}$

6 7 (b)

5	C^b	E^b E^{bb}	G^b
7	D^b	F F^b	A^b
9	E^b	G G^b	B^b
	F^b	A^b A^{bb}	C^b
6	G^b	B^b B^{bb}	D^b
8	A^b	C C^b	E^b
10	B^b	D D^b	F

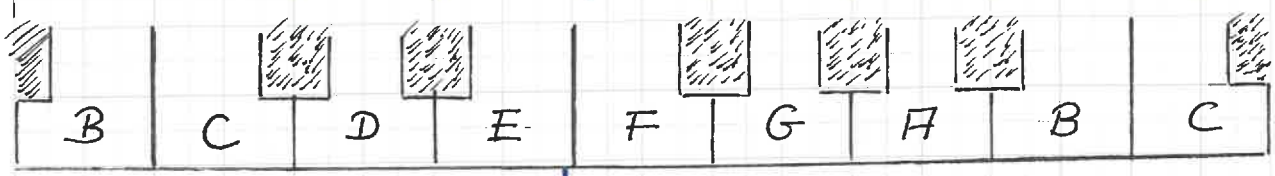
Introduction Sequences for '#' and 'b' for Major Keys

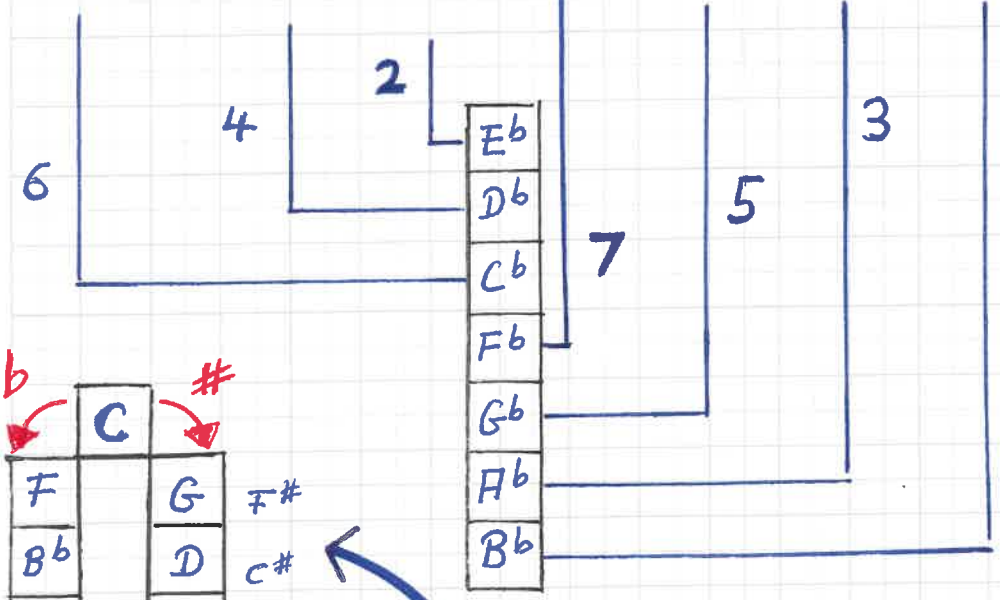
9/17/2024



This diagram shows how the '#' and 'b' are introduced, 1-by-1.

Keyboard







The circle of 5ths introduces '#' in the order $F^{\#}C^{\#}G^{\#}D^{\#}A^{\#}E^{\#}B^{\#}$ and 'b' in the order $B^{b}E^{b}A^{b}D^{b}G^{b}C^{b}F^{b}$.

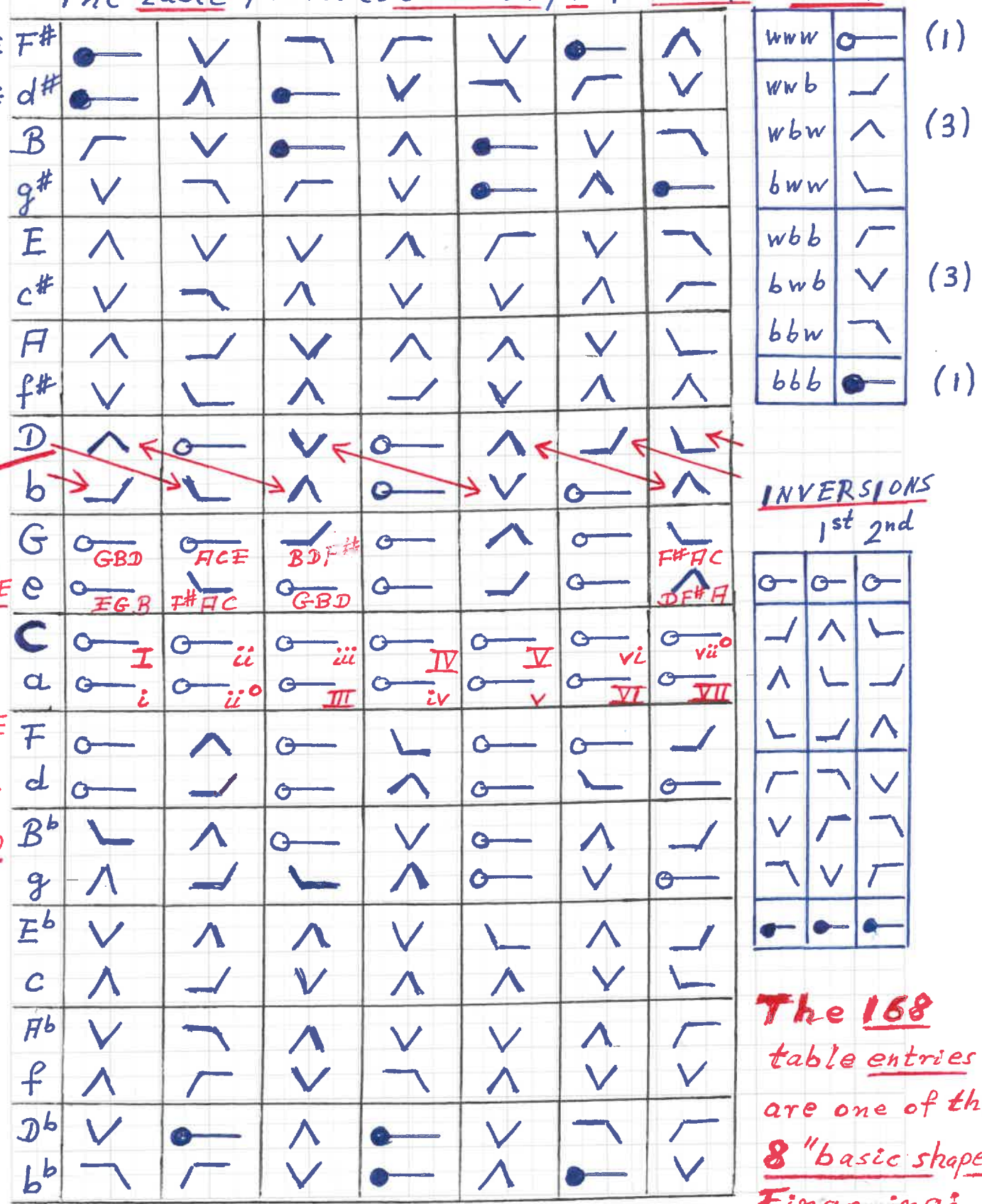
The order of introduction of '#' and 'b' is defined by the index sequence 1-2-3-4-5-6-7.

⇒ INTRODUCTION SEQUENCE(S) SHOULD BE MEMORIZED.

TRIAD "FINGERING SHAPES" FOR MAJOR & MINOR KEYS, WITH IMPLIED INVERSIONS — USING 8 WHITE-BLACK CODES

9/18/2024

The table provides triad shapes for 24 keys.

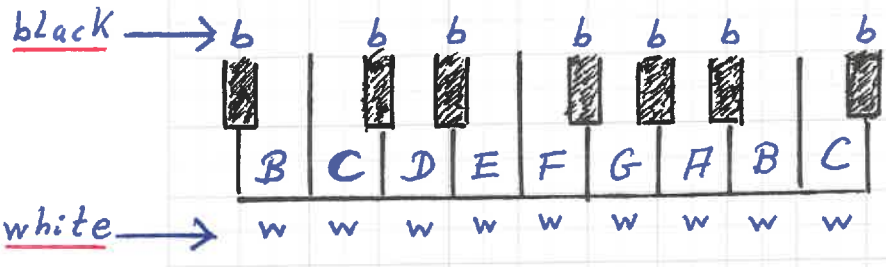


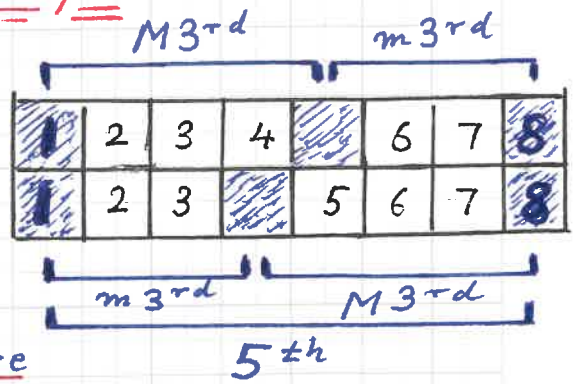
The 168 table entries are one of the 8 "basic shapes"

Fingering: 1-3-5, 5-3-1

• Triads Ordered Based on the Chromatic Scale — 12 Major and 12 Minor Keys

9/19/2024





	Triad	Code
1	C E Eb G	⊙—, ∧
2	Db F Fb Ab	∨, ∨
3	D F# F A	∧, ⊙—
4	Eb G Gb Bb	∨, ●
5	E G# G B	∧, ⊙—
6	F A Ab C	⊙—, ∧
7	F# A# A C#	●, ∨
7	Gb Bb Bbb Db	●, ∨
8	G B Bb D	⊙—, ∧
9	Ab C Cb Eb	∨, ∨
10	A C# C E	∧, ⊙—
11	Bb D Db F	∟, ⌐
12	B D# D F#	⌐, ∟

≙ [7 7]

Black-white (bw) codes for all triads:

www	⊙—	6x
wwb	⌐	1x
wbw	∧	6x
bww	∟	1x
wbb	⌐	1x
bwb	∨	6x
bbw	⌐	1x
bbb	●—	2x

Note. $F^b \hat{=} E$, $C^b \hat{=} B$, $B^{bb} \hat{=} A$

The left table lists the 12×2 = 24 triad "shapes" for all keys — using one of the 8 codes.

The figure above shows that all triads consist of either a major 3rd and a minor 3rd (M3rd m3rd) or a minor 3rd and a major 3rd (m3rd M3rd). The first and last notes of each triad define a 5th, and six half-tones lie between them.

• ONE MUST MEMORIZE AND BE ABLE TO "AUTOMATICALLY FIND AND PLAY" THESE 24 TRIADS — USING ONE OF THE 8 "BASIC SHAPES"

Two-note Patterns/Shapes —

9/21/2024

Chromatic Viewpoint



○ = white (w)
● = black (b)
key

n1 \ n2	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B	C
C	○●	○○	○●	○○	○○	○●	○○	○●	○○	○●	○○	○○
D♭	////	●○	●●	●○	●○	●●	●○	●●	●○	●●	●○	●○
D	○●	////	○●	○○	○○	○●	○○	○●	○○	○●	○○	○○
E♭	●●	●○	////	●○	●○	●●	●○	●●	●○	●●	●○	●○
E	○●	○○	○●	////	○○	○●	○○	○●	○○	○●	○○	○○
F	○●	○○	○●	○○	////	○●	○○	○●	○○	○●	○○	○○
G♭	●●	●●	●●	●○	●○	////	●○	●●	●○	●●	●○	●○
G	○●	○○	○●	○○	○○	○●	////	○●	○○	○●	○○	○○
A♭	●●	●○	●●	●○	●○	●●	●○	////	●○	●●	●○	●○
A	○●	○○	○●	○○	○○	○●	○○	○●	////	○●	○○	○○
B♭	●●	●○	●●	●○	●○	●●	●○	●●	●○	////	●○	●○
B	○●	○○	○●	○○	○○	○●	○○	○●	○○	○●	////	○○

6th
5th
4th
M3rd

M3rd 4th 5th 6th

	2♭	2nd	3♭	3rd	4th	5♭	5th	6♭	6th	7♭	7th	8th
C	○●	○○	○●	○○	○○	○●	○○	○●	○○	○●	○○	○○
D♭	●○	●●	●○	●○	●●	●○	●●	●○	●●	●○	●○	●●
D	○●	○○	○○	○●	○○	○●	○○	○●	○○	○○	○●	○○
E♭	●○	●○	●●	●○	●●	●○	●●	●○	●○	●●	●○	●●
E	○○	○●	○○	○●	○○	○●	○○	○○	○●	○○	○●	○○
F	○●	○○	○●	○○	○●	○○	○○	○●	○○	○●	○○	○○
G♭	●○	●●	●○	●●	●○	●○	●●	●○	●●	●○	●○	●●
G	○●	○○	○●	○○	○○	○●	○○	○●	○○	○○	○●	○○
A♭	●○	●●	●○	●○	●●	●○	●●	●○	●○	●●	●○	●●
A	○●	○○	○○	○●	○○	○●	○○	○○	○●	○○	○●	○○
B♭	●○	●○	●●	●○	●●	●○	●●	●○	●○	●●	●○	●●
B	○○	○●	○○	○●	○○	○○	○●	○○	○●	○○	○●	○○

→ The top table lists the patterns for all note pairs n1 n2, "n1 being left of n2."

→ The left table lists the 12 patterns for "2♭, 2nd, 3♭, 3rd, ..., 8th" for the 12 notes of the chromatic scale.

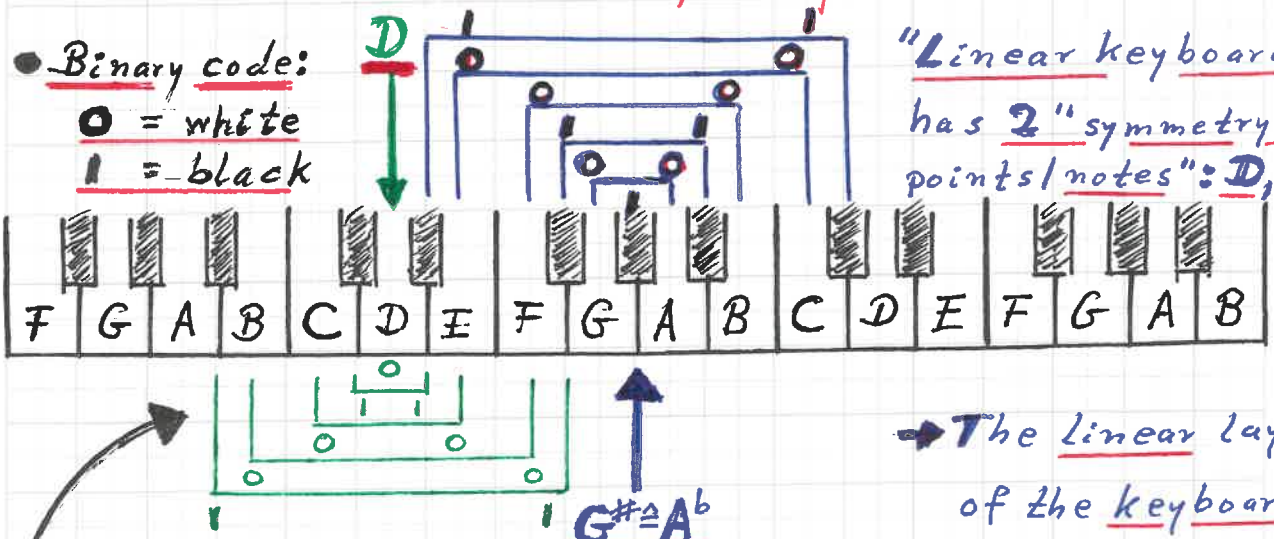
→ MEMORIZE PATTERN

Shapes/patterns for m3rd, M3rd, 4th, 5th, 6th.

Chords and Keyboard Symmetry =

9/22/2024

- Binary code:
  - 0 = white
  - 1 = black



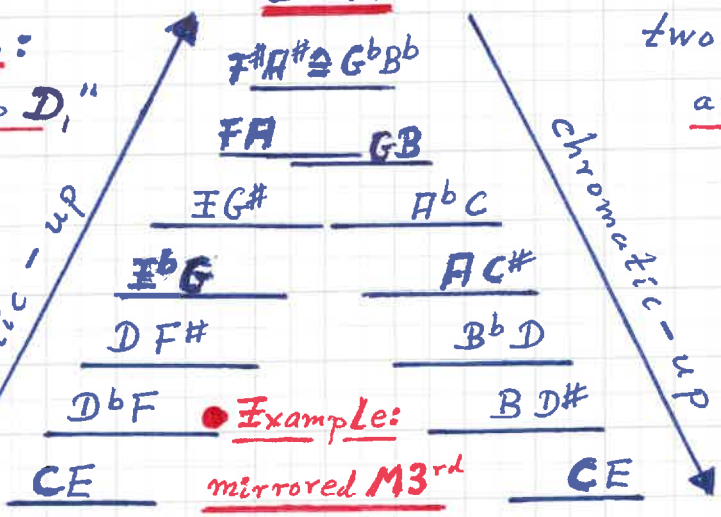
"Linear keyboard" has 2 "symmetry points/notes": D, G#/A♭.

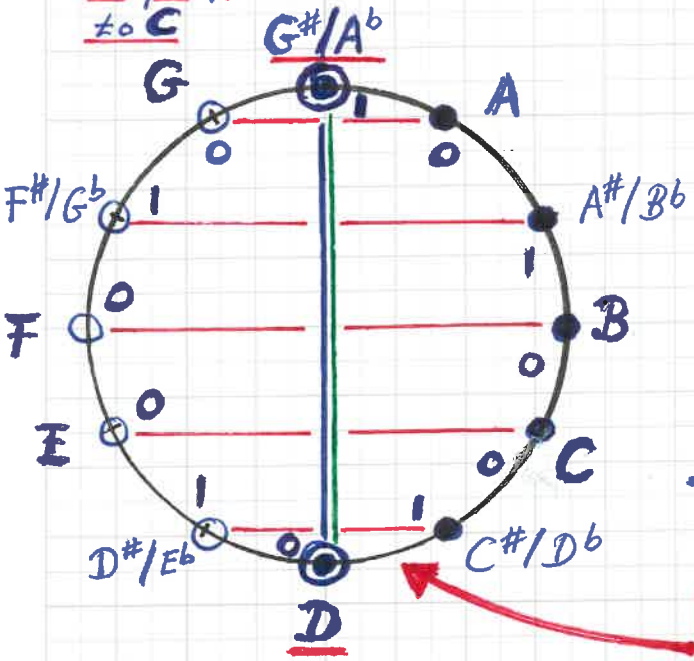
→ The linear layout of the keyboard has two "symmetry keys," at the notes D and G#/A♭. By arranging the notes in a circular chromatic Layout, these two notes define the "symmetry axis" from D to G#/A♭. By drawing this axis vertically, "corresponding reflected mirror images" are the note pairs connected by horizontal edges in the left figure.

→ From note n1 to note n2:
k steps ⇒ from n2 to n1: (12-k) step

Observation:

- "Relative to D,"
  4 half-tone steps to E from C
- "Relative to G#/A♭,"
  8 half-tone steps from E to C





12-tone "chromatic circle" with "symmetry axis from D to G#/A♭. "Symmetry AXIS"!

- CAN ONE USE THIS SYMMETRY, THE "MODULO-12 NATURE" OF SCALES & BINARY CODES FOR CHORDS

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• Sequence/Order of '#' and 'b' "Insertion" into Scales

9/23/2024

➔ The two tables below summarize the "insertion" of '#' and 'b' notes into major scales, using the circle of 5th to define the order of "insertion."

C	O^{C}	O	O	O	O	O^{G}	O	O^{B}	O^{C}	0#
G	O^{G}	O	O	O	O	O	$I^{F\#}$	O^{G}	1#
D	O^{D}	O	I	O	O	O	$I^{C\#}$	O^{D}	2#
A	O^{A}	O	I	O	O	I	$I^{G\#}$	O	3#
E	O^{E}	I	I	O	O	I	$I^{D\#}$	O	4#
B	O^{B}	I	I	O	I	I	$I^{A\#}$	O	5#
F#	$I^{F\#}$	I	I	O	I	I	$O^{E\#}$	$I^{F\#}$	6#
C#	$I^{C\#}$	I	$O^{E\#}$	I	$I^{F\#}$	I	I	$O^{B\#}$	$I^{C\#}$	7#

Black-white patterns for '#' (O = white, I = black).

C	O^{C}	O	O	O^{F}	O	O	O^{B}	O^{C}	0b
F	O^{F}	O	O	$I^{B\flat}$	O	O	O^{E}	O^{F}	1b
B♭	$I^{B\flat}$	O	O	$I^{E\flat}$	O	O	O^{A}	$I^{B\flat}$	2b
E♭	$I^{E\flat}$	O	O	$I^{A\flat}$	$I^{B\flat}$	O	O^{D}	$I^{E\flat}$	3b
A♭	$I^{A\flat}$	I	O	$I^{D\flat}$	I	O	O^{G}	$I^{A\flat}$	4b
D♭	$I^{D\flat}$	I	O	$I^{G\flat}$	I	I	O^{C}	$I^{D\flat}$	5b
G♭	$I^{G\flat}$	I	I	$O^{C\flat}$	I	I	O^{F}	$I^{G\flat}$	6b
C♭	$O^{C\flat}$	I	I	$O^{F\flat}$	I	I	$I^{B\flat}$	$O^{C\flat}$	7b

Black-white patterns for 'b' (O = white, I = black).

EQUIVALENT

Note. The "insertion" of $E^{\#}$, $B^{\#}$, $C^{\flat}$ and $F^{\flat}$ DO NOT lead to the introduction of BLACK keys — since $E^{\#} \hat{=} F$, $B^{\#} \hat{=} C$, $C^{\flat} \hat{=} B$ and $F^{\flat} \hat{=} E$.

➔ One must memorize the two "insertion sequences" $F^{\#}\,C^{\#}\,G^{\#}\,D^{\#}\,A^{\#}\,E^{\#}\,B^{\#}$ and $B^{\flat}\,E^{\flat}\,A^{\flat}\,D^{\flat}\,G^{\flat}\,C^{\flat}\,F^{\flat}$ — and the corresponding black-white patterns and changes on the keyboard.

➔ The tables indicate the "equivalence" of certain keys: $B \hat{=} C^{\flat}$, $F^{\#} \hat{=} G^{\flat}$ and $C^{\#} \hat{=} D^{\flat}$.

Since these keys/scales are "equivalent," the corresponding black-white key patterns are the same.

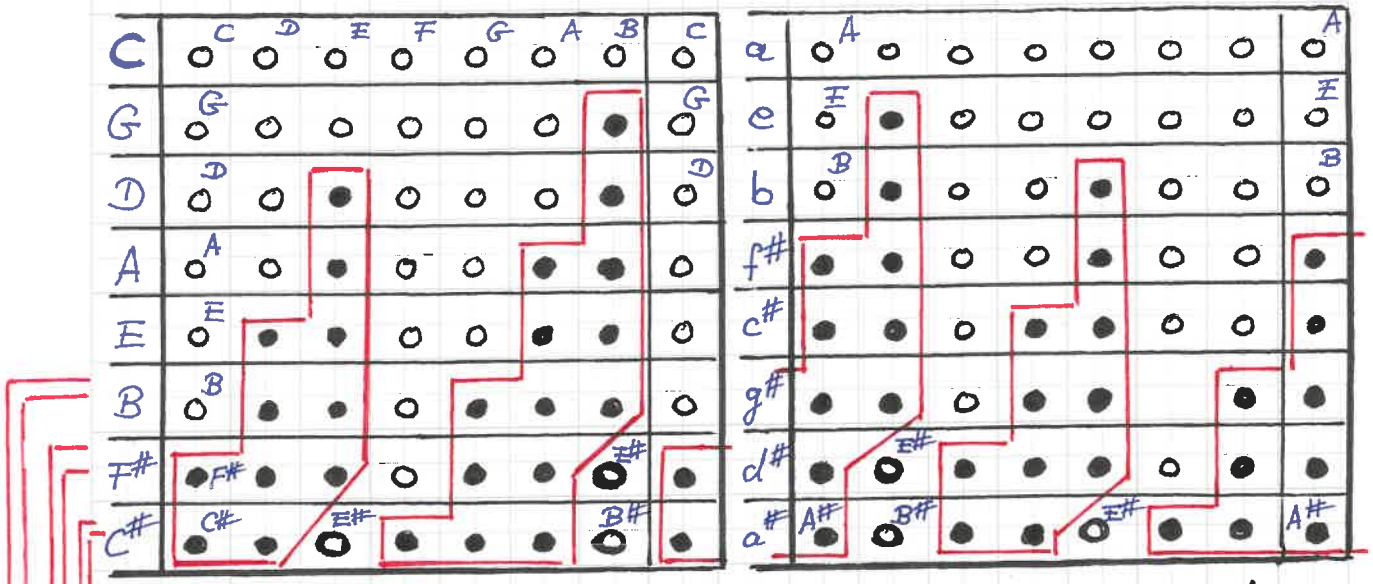
➔ The patterns for B, $F^{\#}$ and $C^{\#}$ ($C^{\flat}$, $G^{\flat}$ and $D^{\flat}$) major include all available black keys.

➔ Key patterns for minor keys are defined by their relative major keys.

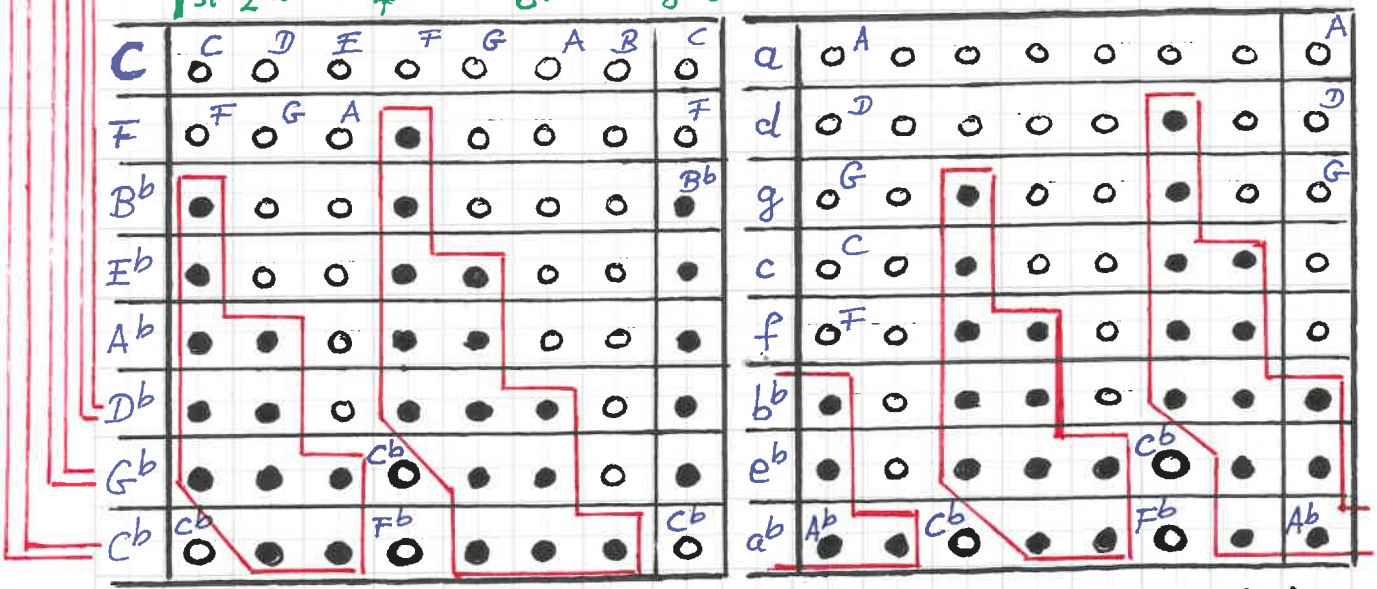
SEQUENCE/ORDER OF '#' AND 'b' "INSERTIONS" INTO SCALES

9/24/2024

→ The tables summarize "insertions" for major and minor scales:



Major and relative natural minor scales — '#'.



EQUIVALENT

Major and relative natural minor scales — 'b'.

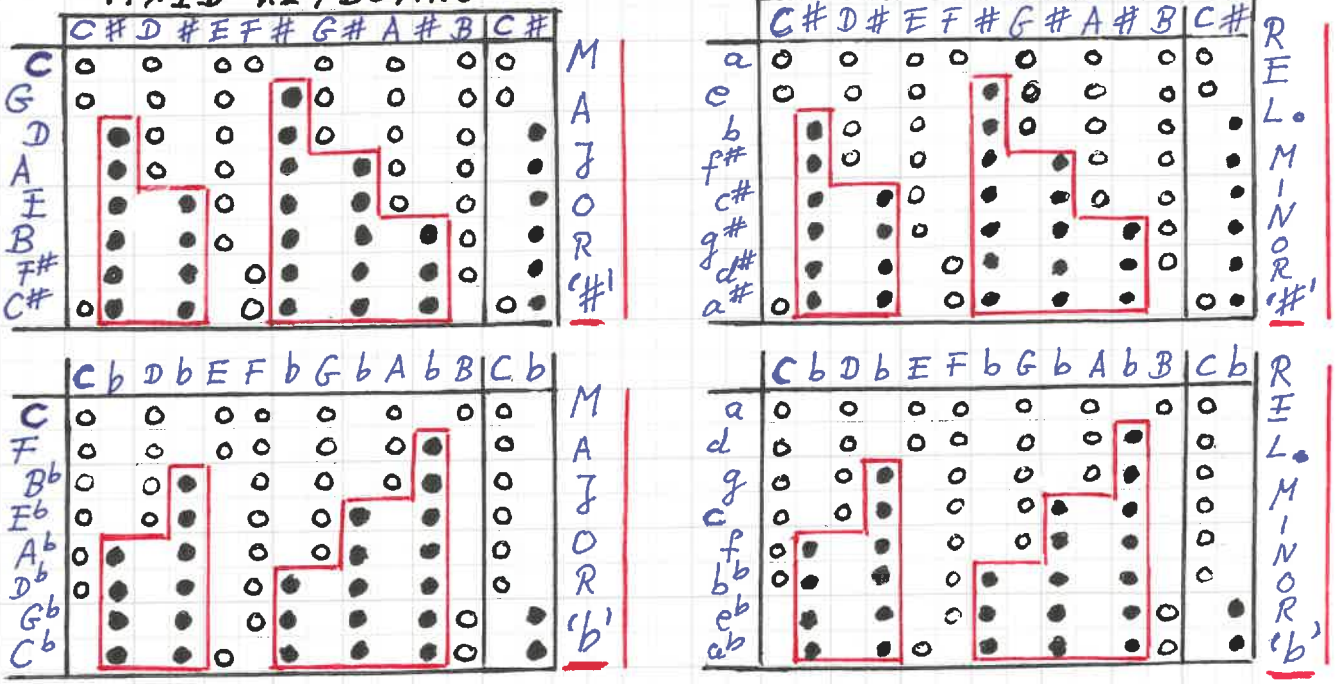
→ The sequences/orders of "insertions" of '#' and 'b' AND THE RESULTING SETS OF BLACK KEYS 'o' to be used on the piano keyboard are highlighted by showing the enclosing polygons (red).

→ **Note.** Since $E^{\#} \hat{=} F$, $B^{\#} \hat{=} C$, $C^{b} \hat{=} B$ and $F^{b} \hat{=} E$, are played with the corresponding white keys 'o'.

SEQUENCE/ORDER OF '#' AND 'b' "INSERTIONS" ON KEYBOARD

9/25/2024

"FIXED KEYBOARD"



→ The tables above illustrate the sequences/orders of "inserting" black keys (●) when using the circle-of-5th system for "inserting" '#' and 'b' into major keys and their relative, natural minor keys.

→ Two patterns stand out, shown via a red polygon(s) enclosing the "inserted" black keys:

1) When inserting '#' according to the circle-of-5th's, the black-key "insertion" order is $F^{\#} - C^{\#} - G^{\#} - D^{\#} - A^{\#}$. The notes $E^{\#} \hat{=} F$ and $B^{\#} \hat{=} C$ have white keys on the keyboard.

2) When "inserting" 'b' according to the circle-of-5th's, the black-key "insertion" order is $B^{b} - E^{b} - A^{b} - D^{b} - G^{b}$. The notes $C^{b} \hat{=} B$ and $F^{b} \hat{=} E$ have white keys on the keyboard.

→ One must only memorize these two "insertion" orders — for '#' and 'b'.

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• Key-inherent Triads Viewed as Variations of 7 Basic Triads

9/26/2024

C		G			D			A			E			B			F#			C#			a			e			b...
I	CEG				#			#		#	#		#	#		#	#	#	#	#	#	#	III						f#...
ii	DFA		#			#			#		#	#		#	#	#	#	#	#	#	#	#	iv				#		c#...
iii	EGB							#			#			#			#	#		#	#	#	v						g#...
IV	FAC	#			#		#		#		#	#		#	#	#	#	#	#	#	#	#	VI				#		d#...
V	GBD							#			#		#	#		#	#		#	#	#	#	VII						a#...
vi	ACE					#			#			#		#	#		#	#	#	#	#	#	i						(left
vii°	BDF			#			#			#		#	#		#	#		#	#	#	#	#	ii°					#	table!
		0#			3#			6#			9#			12#			15#			18#			21#			0#			3#	

numbering for C major

E# ≙ F B# ≙ C

numbering for a minor

The 7 basic triads CEG, DFA, ..., BDF — and their modifications via '#'.

Relative minor triads.

C		F			B♭			E♭			A♭			D♭			G♭			C♭			a			d			g...
I	CEG					b			b			b			b	b	b	b	b	b	b	b	b	III						c...
ii	DFA									b	b		b	b		b	b		b	b	b	b	iv						f...
iii	EGB			b	b		b	b		b	b		b	b	b	b	b	b	b	b	b	b	v					b		b♭...
IV	FAC								b			b			b			b	b	b	b	b	VI						e♭...
V	GBD		b			b			b			b	b	b	b	b	b	b	b	b	b	b	VII				b			a♭...
vi	ACE							b	b		b	b		b	b		b	b	b	b	b	b	i							(left
vii°	BDF	b			b			b			b	b		b	b		b	b		b	b	b	ii°			b				table!)
		0b			3b			6b			9b			12b			15b			18b			21b			0b			3b	

C♭ ≙ B F♭ ≙ E

The 7 basic triads modified via 'b'.

Relative minor triads.

→ One must only MEMORIZE the 7 BASIC TRIADS — their alphabetical representation and Location on the keyboard — and their MODIFICATION by '#', 'b'.

SCALES, CHORDS & INVERSIONS: GOOD METHODS FOR MEMORIZATION

9/27/2024

→ Simple and efficient memorization methods are necessary for quickly recalling notes and keyboard fingerings for playing.
- SCALES of all major and minor keys;
- TRIADS, 7th CHORDS and their INVERSIONS;
- KEY-INHERENT TRIADS for all keys.

→ ONE MUST KNOW THE SEQUENCES/ORDERS THAT DEFINE THE INTRODUCTION/ADDITION OF '#' AND 'b' BASED ON THE CIRCLE-OF-5th.

The two sequences/orders are:

#: $F^{\#}$, $C^{\#}F^{\#}$, $C^{\#}F^{\#}G^{\#}$, $C^{\#}D^{\#}F^{\#}G^{\#}$, $C^{\#}D^{\#}F^{\#}G^{\#}A^{\#}$, $C^{\#}D^{\#}E^{\#}$ (≙F) $F^{\#}G^{\#}A^{\#}$, $C^{\#}D^{\#}E^{\#}$ (≙F) $F^{\#}G^{\#}A^{\#}B^{\#}$ (≙C).

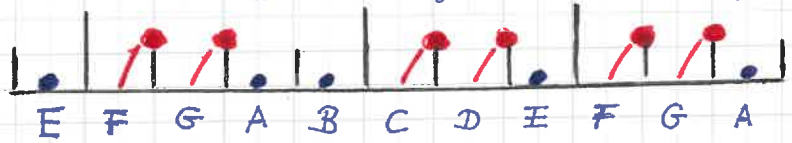
b: B^{b}, $E^{b}B^{b}$, $E^{b}A^{b}B^{b}$, $D^{b}E^{b}A^{b}B^{b}$, $D^{b}E^{b}G^{b}A^{b}B^{b}$, $D^{b}E^{b}G^{b}A^{b}B^{b}C^{b}$ (≙B), $D^{b}E^{b}F^{b}$ (≙E) $G^{b}A^{b}B^{b}C^{b}$ (≙B) (C^{b} (≙B) $D^{b}E^{b}G^{b}A^{b}B^{b}$, C^{b} (≙B) $D^{b}E^{b}F^{b}$ (≙E) $G^{b}A^{b}B^{b}$).

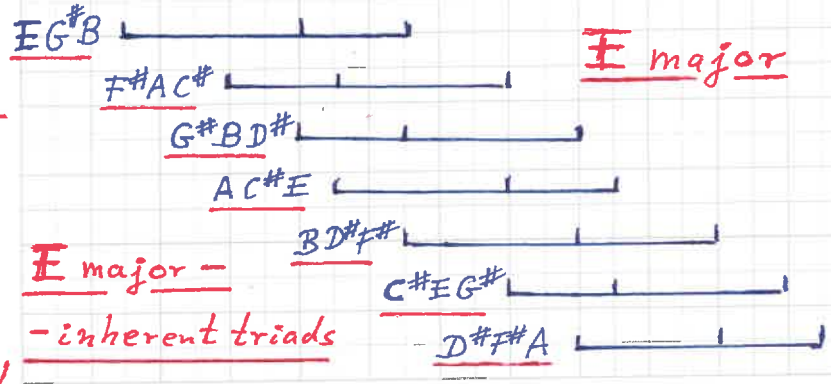
•Playing all SCALES ←

These two sequences/orders for '#' and 'b' use ALPHABETICAL ORDER, to reflect the corresponding left-to-right keyboard locations.

•Principle for Playing all 7 Key-inherent Triads for all Keys

- with Inversions implied





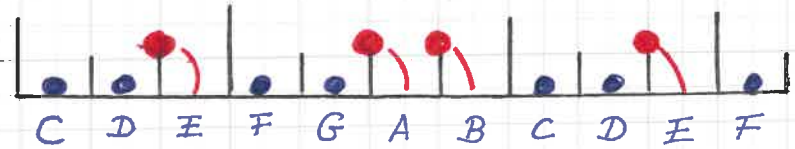
E major

E major-inherent triads

→ The left figure shows an example for the 7 triads being the E major-inherent triads. The 7 chords are based on the 7 basic chords EGB, FAC, ..., DFA with properly added '#'.

• SCALES, CHORDS & INVERSIONS: GOOD METHODS FOR MEMORIZATION

9/27/2024



→ The Left figure shows another example for the 7 triads being the c-minor-inherent triads. The 7 triads are based on the 7 basic triads CEG, DFA, ..., BDF with properly added 'b'.

C minor

CDE^b

DFA^b

E^bGB^b

FA^bC

GB^bD

A^bCE^b

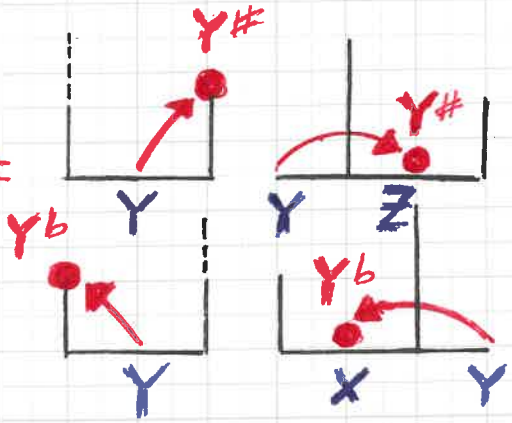
B^bDF

C minor-inherent triads



→ The left figure shows an "extreme" example for the definition of the 7 triads being the C# major-inherent triads. Again, these triads are based on the 7 basic triads and – with proper use of all '#' – become $C^\#E^\#G^\#$, $D^\#F^\#A^\#$, $E^\#G^\#B^\#$, $F^\#A^\#C^\#$, $G^\#B^\#D^\#$, $A^\#C^\#E^\#$, $B^\#D^\#F^\#$.

• Rules for Playing all Scales



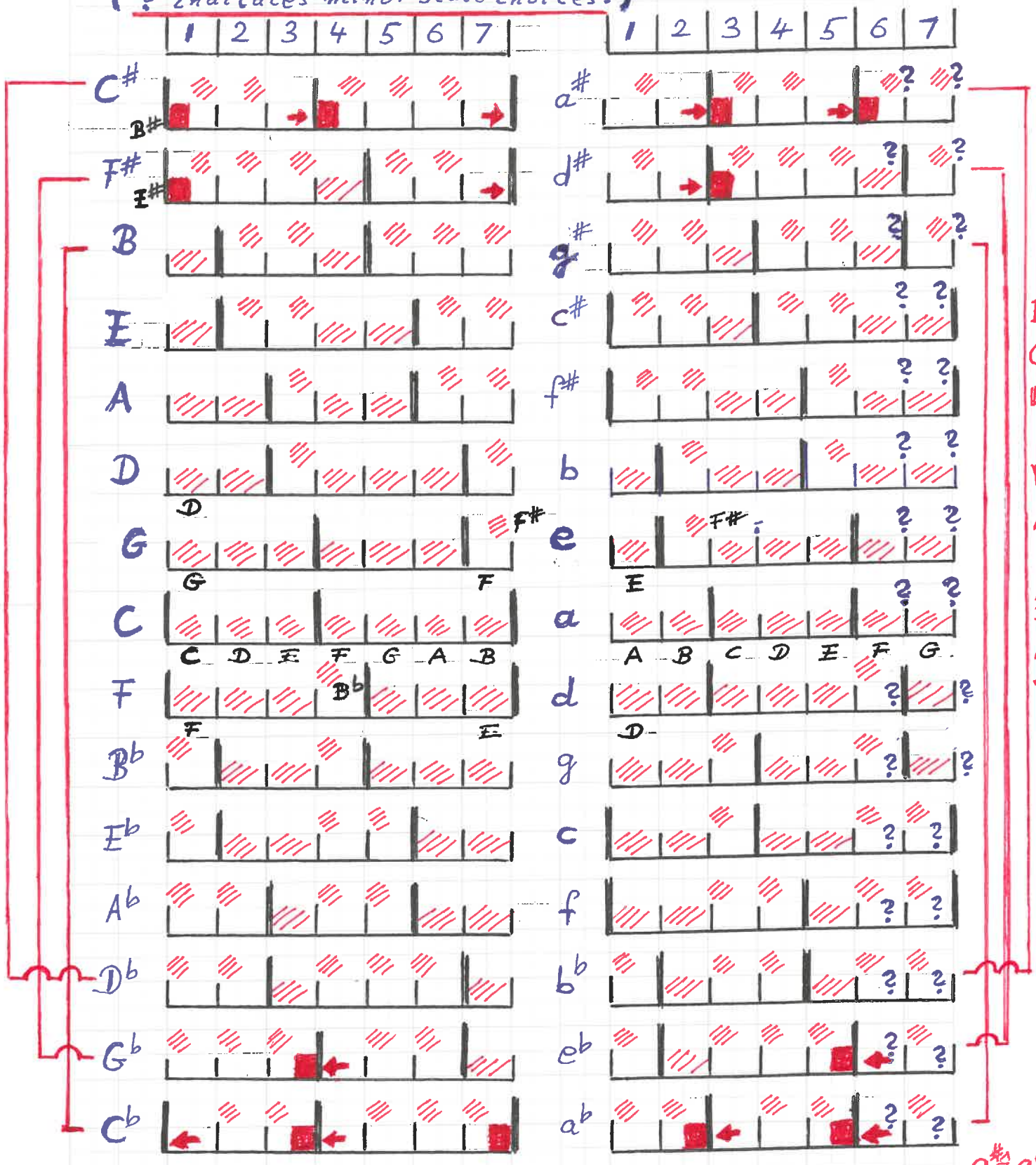
→ The four left figures explain the representation of the effects (on the keyboard) when MOVING TO THE RIGHT as a consequence of changing a note Y to $Y^\#$ AND MOVING TO THE LEFT as a consequence of changing a note Y to Y^b. The two right figures illustrate the cases of $Y^\# \hat{=} Z$ (white key) and $Y^b \hat{=} X$ (white key).

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Key/Fingering Charts to Memorize

9/28/2024

The illustrations show the key patterns of all scales.
('?' indicates minor scale choices.)



$B \triangleq C^{\flat}$
$F^{\#} \triangleq G^{\flat}$
$C^{\#} \triangleq D^{\flat}$

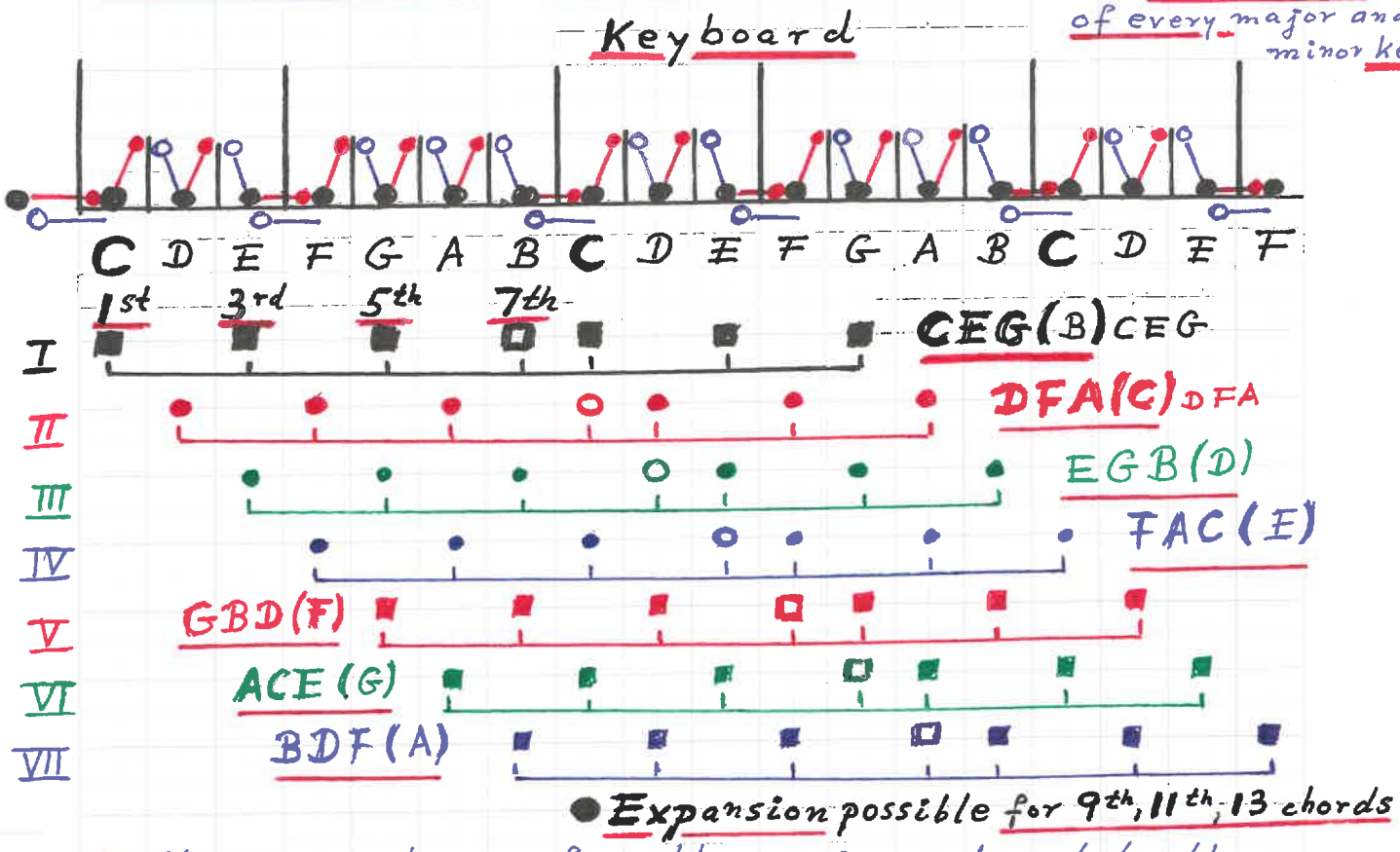
$g^{\#} \triangleq a^{\flat}$
$d^{\#} \triangleq e^{\flat}$
$a^{\#} \triangleq b^{\flat}$

→ One can think of all patterns as patterns of the white keys for C, D, E, F, G, A, B — where BLACK KEYS ARE VIEWED AS EXTENSIONS OF WHITE KEYS.

• Scales, Triads, 7th Chords, Inversions, Key-inherent Chords, ...

9/29/2024

• One must know the '#'s and 'b's of every major and minor key.



• Expansion possible for 9th, 11th, 13 chords

→ Memorization of patterns is reduced to the seven notes C, D, E, F, G, A, B (white keys only).

→ Each of the letters C, D, ..., B simultaneously represents its "three harmonic definitions, i.e., $C \hat{=} \{C^b, C, C^\#\}, \ldots, B \hat{=} \{B^b, B, B^\#\}$.

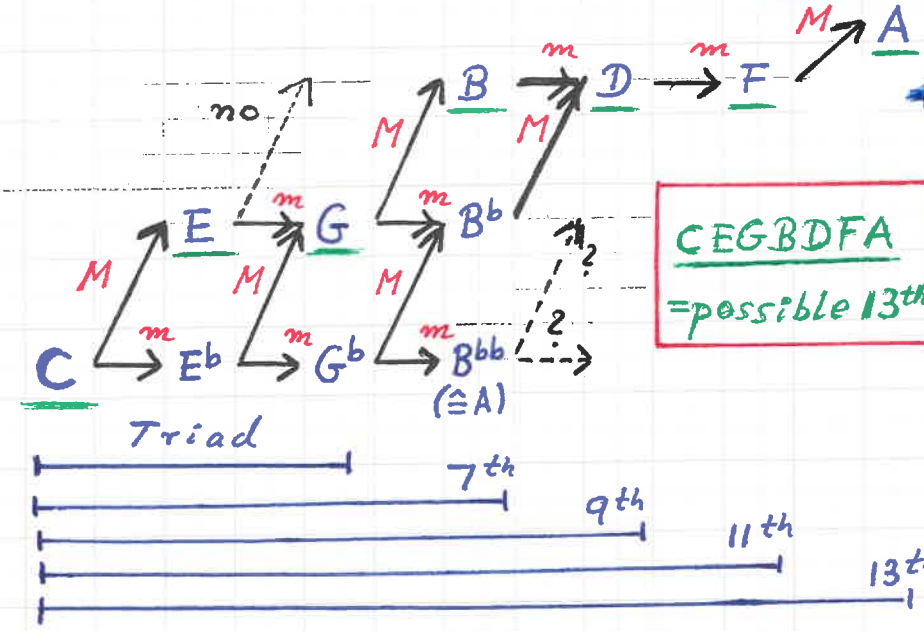
→ Considering necessary modifications of C, D, E, F, G, A, B via '#' or 'b' required by a specific key, are indicated in the keyboard illustration by / and — for '#' and 🔍 and o— for 'b' symbols, implying black keys.

→ The seven key-inherent triads and 7th chords (I – VII) are shown under the keyboard. For each of the shown triads and 7th chords, the 1st, 3rd, 5th and 7th notes are indicated — repeating the 1st, 3rd and 5th notes one octave higher for inversions: An example is CEGB $\xrightarrow{1}$ EGBC $\xrightarrow{2}$ GBCE $\xrightarrow{3}$ BCEG.

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Representation of 7th, 9th, 11th, 13th Chords 9/30/2024

➡ One can view these complex "jazz chords" as "stacks of 3rds." One stacks major and minor 3rds. In the following, a major 3rd chord is called M3rd or M3 or simply M; a minor 3rd chord is called m3rd or m3 or simply m. One can use a graph representation to capture all possible note stacks.



CEGBDFA
=possible 13th

➡ Using C as root note, for example, one can use the directed graph shown here to construct all possible chords obtained by stacking M3rds and m3rds. The figure indicates that some possible choices are (usually) not considered ("no") and that it is not immediately clear whether certain theoretical possibilities are "harmonically meaningful" ("?").

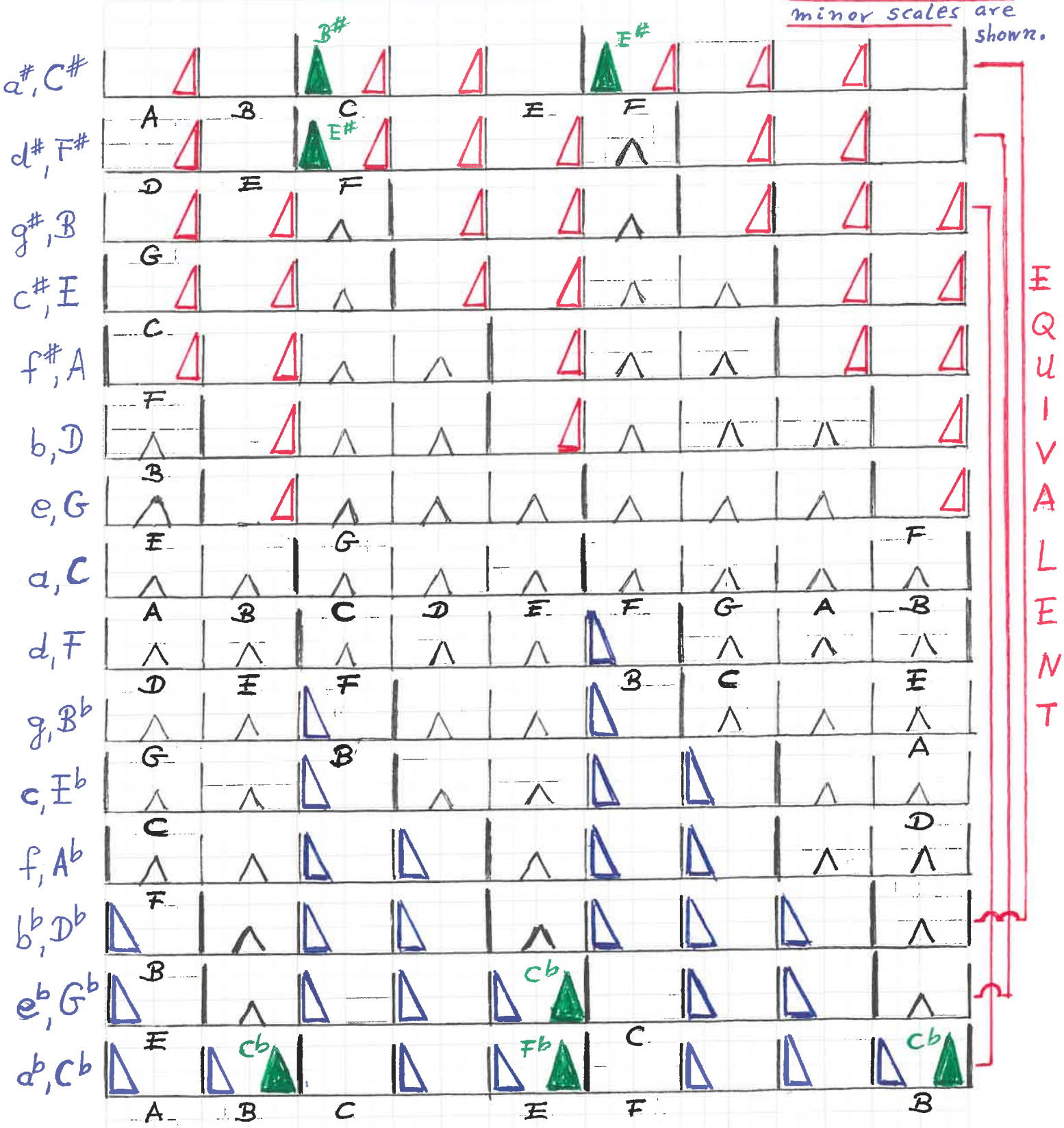
➡ The possible paths in the directed graph represent possible chords; one often does not (have to) play all notes of very complex chords. The underlined notes in the graph highlight the fact that all notes of a key are a 13th.

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• Major and Minor Scales — Represent with White Keys only!

9/30/2024

• Major and relative natural minor scales are shown.



• Principle: All major and minor scales are "considering" only the notes C, D, E, F, G, A, B. Key-defined '#' and 'b' are "thought of" as "white-key-associated parts." The figure uses four colored symbols to indicate keys to be played.

Scales and Chords – Various Thoughts

10/1/2024

Piano Keyboard – Numbered Keys

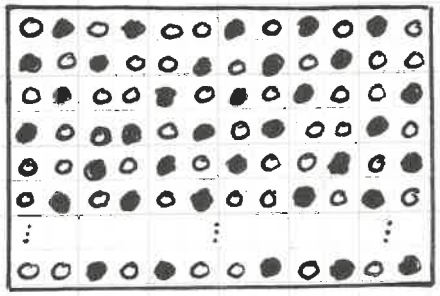
No. of keys: $3+7\cdot 12+1 = 88$



Chromatic Scales and Use of White / Black Keys:

1	2	3	4	5	6	7	8	9	10	11	12
C	C#	D	D#	E	F	F#	G	G#	A	A#	B
C#	D	D#	E	F	F#	G	G#	A	A#	B	C
D	D#	E	F	F#	G	G#	A	A#	B	C	C#
D#	E	F	F#	G	G#	A	A#	B	C	C#	D
E	F	F#	G	G#	A	A#	B	C	C#	D	D#
F	F#	G	G#	A	A#	B	C	C#	D	D#	E
F#	G	G#	A	A#	B	C	...			...	F
⋮											⋮
B	C	C#	D	D#	E	F	F#	G	G#	A	A#

The left table lists the "12 chromatic scales" – using as 1st note C, C#, D, ..., A, A#, B. The resulting 12 scales lead to a symmetric 12×12 "binary matrix" representing the corresponding use of white (○) and black (●) keys on the keyboard. The left figure illustrates this 12×12 "matrix of patterns."



Chord Notes Viewed as "Sets of Notes":

Often / usually, the lowest (highest) note played serves as bass (melody) note with its special function. Since the "notes in a set" do not reflect specific keys to be played, one can "pick" notes from the set to define a bass note or define inversions – "spreading out notes into both hands."

C used as bass note (left); "4 inversions" from set {C, E, G, B♭}

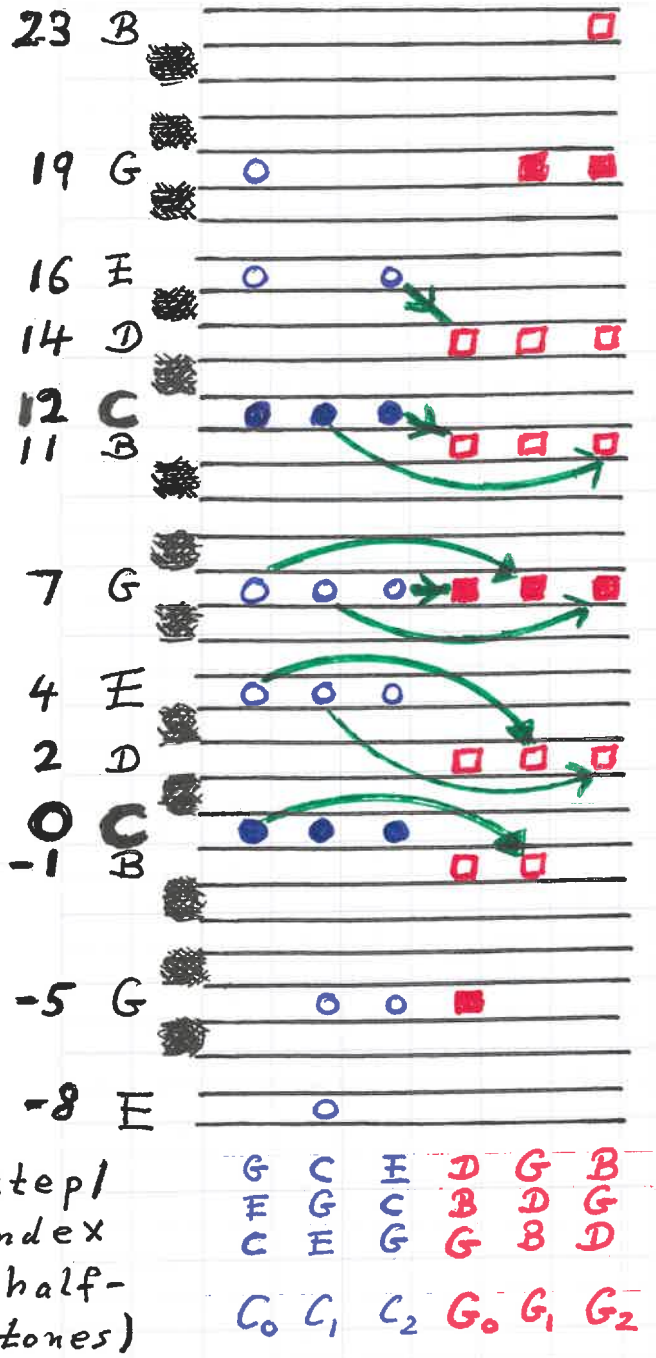
C	E	G	B♭
C	E	B♭	G
C	G	E	B♭
C	G	B♭	E
C	B♭	E	G
C	B♭	G	E

C	E	G	B
E	G	B	C
G	B	C	E
B	C	E	G

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Voice Leading & Optimization

10/11/2024



It is possible to use a simple mathematical approach to determine a "good voicing" when establishing a chord progression that moves from chord to chord (including chord inversions). The left figure shows C and G major chords in root and 1st and 2nd inversion positions, called C_0, C_1, C_2 and G_0, G_1, G_2, respectively. All chords are shown twice, shown "in two different octaves." The goal is to determine the pair(s) of a C and G major chord(s) defining the "smoothest" voicing(s) — indicated by the green arrows in the figure.

The figure shows the "keyboard vertically-1" uses one line per note in the 12-tone chromatic system, and uses integers as "indices" for the lines (with 0 being associated with [one of] the C notes):

$$C_0=\binom{7}{4}_0+i\binom{12}{12}_{12},\ C_1=\binom{12}{7}_4+i\binom{12}{12}_{12},\ C_2=\binom{16}{12}_7+i\binom{12}{12}_{12},\ i=\ldots,-1,0,1,\ldots,$$

and

• Voice Leading & Optimization

10/11/2024

$G_0 = \begin{pmatrix}14\\11\\7\end{pmatrix} + j\begin{pmatrix}12\\12\\12\end{pmatrix}$, $G_1 = \begin{pmatrix}19\\14\\11\end{pmatrix} + j\begin{pmatrix}12\\12\\12\end{pmatrix}$, $G_2 = \begin{pmatrix}23\\19\\14\end{pmatrix} + j\begin{pmatrix}12\\12\\12\end{pmatrix}$, $j = \ldots -1, 0, 1, \ldots$

represent "all possible" C_0, C_1, C_2 and G_0, G_1, G_2 chords in all octaves available on the keyboard (and "beyond"). For example, the goal could be the "smoothest" moves/progressions (I-V) from $C_0 = \begin{pmatrix}7\\4\\0\end{pmatrix}$ to one of the G chords; from $C_1 = \begin{pmatrix}12\\7\\4\end{pmatrix}$ to one of the G chords; and from $C_2 = \begin{pmatrix}16\\12\\7\end{pmatrix}$ to one of the G chords. Therefore, one must define a simple "distance" for two chords — two triads in this scenario — to determine a pair of C and G major triads that are "closest" to each other, based on individual "note-to-note distances." For two triads $N_1 N_2 N_3$ and $n_1 n_2 n_3$ (with N_i and n_i being their notes) one can define a "distance" as the sum of the absolute values of corresponding note-pair distances, i.e., $\text{distance} = |N_1 - n_1| + |N_2 - n_2| + |N_3 - n_3|$, where the values of N_i and n_i are the note-associated integers shown in the figure on the previous page. For the three chords C_I, $I = 0, 1, 2$, and G_J, $J = 0, 1, 2$, one can thus compute $3 \times 3 = 9$ "distance" values $d_{I,J}$ — that define the minimal-distance chord G_J for a given chord C_I. Thus, we must calculate the values of $d_{0,J}$, $d_{1,J}$ and $d_{2,J}$ $(J = 0, 1, 2)$ for C_0, C_1, C_2 as defined above.

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• Note. For a given chord (C_0 or C_1 or C_2) one must first determine the closest chord (G_0, G_1, G_2) — from the set of all the considered G chords on the keyboard — to be able to determine the smallest values for $d_{0,0} = dist(C_0, G_0)$, $d_{0,1} = dist(C_0, G_1)$, ..., $d_{2,2} = dist(C_2, G_2)$. Assuming that these closest G chords are known on the keyboard for the specific chords C_0, C_1 and C_2, one can calculate the values of $d_{0,0}, \dots, d_{2,2}$ as described in the following.

$\Rightarrow$ Chord-to-chord distances $d_{I,J}$, $I,J = 0,1,2$:

$C_0 = \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix}$, $C_1 = \begin{pmatrix} 12 \\ 7 \\ 4 \end{pmatrix}$, $C_2 = \begin{pmatrix} 16 \\ 12 \\ 7 \end{pmatrix}$; $G_0 = \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix}$, $G_1 = \begin{pmatrix} 7 \\ 2 \\ -1 \end{pmatrix}$, $G_2 = \begin{pmatrix} 11 \\ 7 \\ 2 \end{pmatrix}$

$\Rightarrow d_{0,0} = |0+5| + |4+1| + |7-2| = 5+5+5 = 15$

$d_{0,1} = |0+1| + |4-2| + |7-7| = 1+2+0 = \mathbf{3}$ MIN

$d_{0,2} = |0-2| + |4-7| + |7-11| = 2+3+4 = 9$

$G_0 = \begin{pmatrix} 14 \\ 11 \\ 7 \end{pmatrix}$

$d_{1,0} = |4-7| + |7-11| + |12-14| = 3+4+2 = 9$

$d_{1,1} = |4+1| + |7-2| + |12-7| = 5+5+5 = 15$

$d_{1,2} = |4-2| + |7-7| + |12-11| = 2+0+1 = \mathbf{3}$ MIN

$d_{2,0} = |7-7| + |12-11| + |16-14| = 0+1+2 = \mathbf{3}$ MIN

$G_1 = \begin{pmatrix} 19 \\ 14 \\ 11 \end{pmatrix}$

$d_{2,1} = |7-11| + |12-14| + |16-19| = 4+2+3 = 9$

$d_{2,2} = |7-2| + |12-7| + |16-11| = 5+5+5 = 15$

$\Rightarrow \mathcal{D} = \begin{bmatrix} d_{0,0} & d_{0,1} & d_{0,2} \\ d_{1,0} & d_{1,1} & d_{1,2} \\ d_{2,0} & d_{2,1} & d_{2,2} \end{bmatrix} = \begin{bmatrix} 15 & 3 & 9 \\ 9 & 15 & 3 \\ 3 & 9 & 15 \end{bmatrix}$.

OPTIMAL:

C_0G_1: (0,4,7)(-1,2,7);

C_1G_2: (4,7,12)(2,7,11);

C_2G_0: (7,12,16)(7,11,14).

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One can summarize this result as follows:

1) Given the chord CEG "on steps 0, 4 and 7," the closest G triad is BDG "on steps -1, 2 and 7."

2) Given the chord EGC "on steps 4, 7 and 12," the closest G triad is DGB "on steps 2, 7 and 11."

3) Given the chord GCE "on steps 7, 12 and 16," the closest G chord is GBD "on steps 7, 11 and 14."

AND: The three calculated smallest-chord-distance pairs CEG-BDG, EGC-DGB and GCE-GBD all have the same minimal chord-to-chord (here, triad-to-triad) distance: **3**. In other words, these three pairs have a "summed up" note-to-note distance of only **3** halftones.

• As mentioned, for a given triad $N_1 N_2 N_3$ one must, in a "pre-step," determine the closest triad (on the keyboard) $n_1 n_2 n_3$ from the set of "octave-shifted versions" of $n_1 n_2 n_3$. One has the "fixed" triad $\begin{pmatrix} N_3 \\ N_2 \\ N_1 \end{pmatrix}$ and one must determine the/a "variable" triad $\begin{pmatrix} n_3 \\ n_2 \\ n_1 \end{pmatrix} + j \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$ that has minimal distance from the "fixed" triad. Again, N_i and n_i are integers (representing notes), and the value of j is the value to be optimally calculated. ONE CAN VIEW THE AVERAGE $(N_1+N_2+N_3)/3$ AS "THE MIDPOINT" OF THE FIXED TRIAD $\begin{pmatrix} N_3 \\ N_2 \\ N_1 \end{pmatrix}$. THE AVERAGE OF THE VARIABLE TRIAD $\begin{pmatrix} n_3+j\,12 \\ n_2+j\,12 \\ n_1+j\,12 \end{pmatrix}$ IS "THE MIDPOINT" $(n_1+n_2+n_3)/3 + 12j$. ONE MUST DETERMINE j.

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Thus, one determines the/an optimal value for j by minimizing the chord-to-chord — i.e., triad-to-triad — distance, given as $(N_1+N_2+N_3)/3 - \left((n_1+n_2+n_3)/3 + 12j\right) = (M-m) - 12j$. Here, $M=(N_1+N_2+N_3)/3$, $m=(n_1+n_2+n_3)/3$ and $j=\ldots,-1,0,1,\ldots$. Therefore, the goal is to "move" or "shift" the variable chord by an integer-multiple of 12 (12 halftones). MORE PRECISELY, ONE MUST DETERMINE THE INTEGER j FROM THE FINITE SET $\{\ldots,-2,-1,0,1,2,\ldots\}$ THAT MINIMIZES THE ABSOLUTE VALUE OF $(M-m)-12j$, i.e., THE VALUE OF $|(M-m)-12j| = |\Delta-12j|$, where $\Delta = M-m$ is the chords' midpoint distance.

Ex.: fixed chord $=\begin{pmatrix}7\\4\\0\end{pmatrix}$; variable chord $=\begin{pmatrix}14\\11\\7\end{pmatrix}+j\begin{pmatrix}12\\12\\12\end{pmatrix}$

$\Rightarrow (N_1,N_2,N_3)=(0,4,7)\,;\,(7+12j,\,11+12j,\,14+12j)$

$\Rightarrow M=(0+4+7)/3=11/3\,;\; m=32/3$

$\Rightarrow \Delta = 11/3 - 32/3 = -21/3 = -7$

"Shifting Go chord on keyboard" ⇒

j	-2	-1	0	1
$\lvert\Delta-12j\rvert$	$\lvert-7+24\rvert = 17$	$\lvert-7+12\rvert = 5$	$\lvert-7-0\rvert = 7$	$\lvert-7-12\rvert = 19$

OPTIMAL: $-j=-1$

⇒ Given the C_0 chord $\begin{pmatrix}7\\4\\0\end{pmatrix} \hat{=} \begin{pmatrix}G\\E\\C\end{pmatrix}$, the closest G_0 chord $\begin{pmatrix}14\\11\\7\end{pmatrix}+j\begin{pmatrix}12\\12\\12\end{pmatrix}$ is the chord $\begin{pmatrix}14\\11\\7\end{pmatrix}-1\begin{pmatrix}12\\12\\12\end{pmatrix}=\begin{pmatrix}2\\-1\\-5\end{pmatrix}\hat{=}\begin{pmatrix}D\\B\\G\end{pmatrix}$.

⇒ The resulting distance between C_0 and G_0 is therefore $d_{0,0}=|0+5|+|4+1|+|7-2|=15$.

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• **Note.** This "mathematically formalized" method for shifting a given "variable chord" by octaves on the keyboard is practically simple. For example, the 88-key keyboard can have the associated halftone numbering of keys as $A0 = -39, \ldots, C1 = -36, \ldots, C2 = -24, \ldots, C3 = -12, \ldots, C4 = 0, \ldots, C5 = 12, \ldots, C6 = 24, \ldots, C7 = 36, \ldots, C8 = 48$. Indeed, this numbering/indexing of all halftones refers to 88 notes: $48 - (-39) + 1 = 88$. Since the keyboard has only seven complete octaves, one can "easily see" the chord in root, 1st inversion or 2nd inversion position that is closest to a given chord — at least for simple chord pairs like triad pairs.

• **Note.** In order to calculate a "distance" for two chords ch1 and ch2 that do not consist of the same number of notes, one must generalize the concepts described above. For example, we use a 7th chord and a triad:

$$ch1 = (C_0^7) \begin{pmatrix} 10 \\ 7 \\ 4 \\ 0 \end{pmatrix} \hat{=} \begin{pmatrix} B^b \\ G \\ E \\ C \end{pmatrix}; \quad ch2 = (F_0) \begin{pmatrix} 12 \\ 9 \\ 5 \end{pmatrix} \hat{=} \begin{pmatrix} C \\ A \\ F \end{pmatrix}.$$

One can compute a distance d_{12} "from ch1 to ch2" and a distance d_{21} "from ch2 to ch1" by determining the closest note in ch2 (ch1) for a note in ch1 (ch2). Using the steps, one obtains $d_{12} = (5+1+2+1)/4 = 9/4$ and $d_{21} = (1+1+2)/3 = 4/3$ — normalizing by the nos. of notes.

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Thus, one could possibly use the (average) value $d = (d_{12} + d_{21})/2 = (9/4 + 4/3)/2 = 43/24$ as chord-to-chord distance, based on the individual chordnote-to-chordnote distances. Here, the value $43/24$ can be interpreted as an average halftone distance of less than 2.

• SMOOTHEST PROGRESSION OF THE EIGHT TRIADS DEFINING PACHELBEL'S CANON. This progression in the key of D major uses the progression I-V-vi-iii-IV-I-IV-V. The triads in root position are given as $DF^{\#}A$, $AC^{\#}E$, $BDF^{\#}$, $F^{\#}AC^{\#}$, GBD, $DF^{\#}A$, GBD and $AC^{\#}E$. One can now ask the following "combinatorial optimization question": Considering this chord progression, what is the optimal sequence of positions (root, 1st inversion or 2nd inversion positions) to be used when playing this progression to make it "sound optimally smooth in a harmonic sense"? Geometrically, one can understand the three positions of a triad as three points in 3D space (using as coordinates the halftone numbers/indices used above). Of all the combinatorically possible ways to establish triad positions (represented via points in 3D space) one must determine the "smoothest polyline in 3D space" connecting 8 "triad points." ⇒ OPTIMAL VOICING!

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One can view the problem of determining an optimal voicing of the chord/triad sequence I-V-vi-iii-IV-I-IV-V in the key of D major in Pachelbel's canon as a problem of determining the best positions (root, 1st inversion or 2nd inversion positions) for each of the 8 major and minor chords/triads when played in the defined order/sequence. When ignoring the specific octaves to be used for playing the triads in, the "position sets" for the triads involved are:

$$I=\left\{\begin{pmatrix}A\\F\#\\D\end{pmatrix},\begin{pmatrix}D\\A\\F\#\end{pmatrix},\begin{pmatrix}F\#\\D\\A\end{pmatrix}\right\};\quad V=\left\{\begin{pmatrix}E\\C\#\\A\end{pmatrix},\begin{pmatrix}A\\E\\C\#\end{pmatrix},\begin{pmatrix}C\#\\A\\E\end{pmatrix}\right\};$$

$$vi=\left\{\begin{pmatrix}F\#\\D\\B\end{pmatrix},\begin{pmatrix}B\\F\#\\D\end{pmatrix},\begin{pmatrix}D\\B\\F\#\end{pmatrix}\right\};\quad iii=\left\{\begin{pmatrix}C\#\\A\\F\#\end{pmatrix},\begin{pmatrix}F\#\\C\#\\A\end{pmatrix},\begin{pmatrix}A\\F\#\\C\#\end{pmatrix}\right\};$$

$$IV=\left\{\begin{pmatrix}D\\B\\G\end{pmatrix},\begin{pmatrix}G\\D\\B\end{pmatrix},\begin{pmatrix}B\\G\\D\end{pmatrix}\right\};\quad I=\ldots;\quad IV=\ldots;\quad V=\ldots\ .$$

$$\Rightarrow I=\{D_0,D_1,D_2\};\ V=\{A_0,A_1,A_2\};\ vi=\{b_0,b_1,b_2\};$$

$$iii=\{f\#_0,f\#_1,f\#_2\};\ IV=\{G_0,G_1,G_2\};\ I=\ldots;\ IV=\ldots;\ V=\ldots\ .$$

Further, one can set $D=D4\hat{=}2$ — and associate integers/indices with the notes of all triads (using, as done above, the definition $C4\hat{=}0$). Thus, all triads can be represented via integer vectors:

$$I=\left\{\begin{pmatrix}9\\6\\2\end{pmatrix},\begin{pmatrix}14\\9\\6\end{pmatrix},\begin{pmatrix}18\\14\\9\end{pmatrix}\right\};\quad V=\left\{\begin{pmatrix}16\\13\\9\end{pmatrix},\begin{pmatrix}21\\16\\13\end{pmatrix},\begin{pmatrix}25\\21\\16\end{pmatrix}\right\};$$

$$vi=\left\{\begin{pmatrix}18\\14\\11\end{pmatrix},\begin{pmatrix}23\\18\\14\end{pmatrix},\begin{pmatrix}26\\23\\18\end{pmatrix}\right\};\quad iii=\left\{\begin{pmatrix}13\\9\\6\end{pmatrix},\begin{pmatrix}18\\13\\9\end{pmatrix},\begin{pmatrix}21\\18\\13\end{pmatrix}\right\};$$

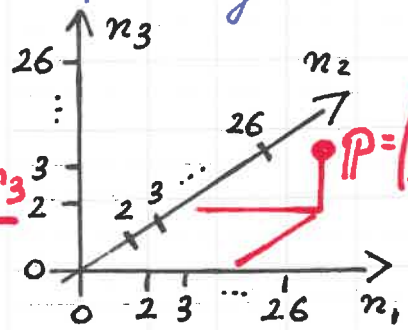
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$$\mathrm{IV} = \left\{ \begin{pmatrix} 14 \\ 11 \\ 7 \end{pmatrix}_1, \begin{pmatrix} 19 \\ 14 \\ 11 \end{pmatrix}_1, \begin{pmatrix} 23 \\ 19 \\ 14 \end{pmatrix} \right\}; \; \mathrm{I} = \ldots; \; \mathrm{IV} = \ldots; \; \mathrm{V} = \ldots \; .$$

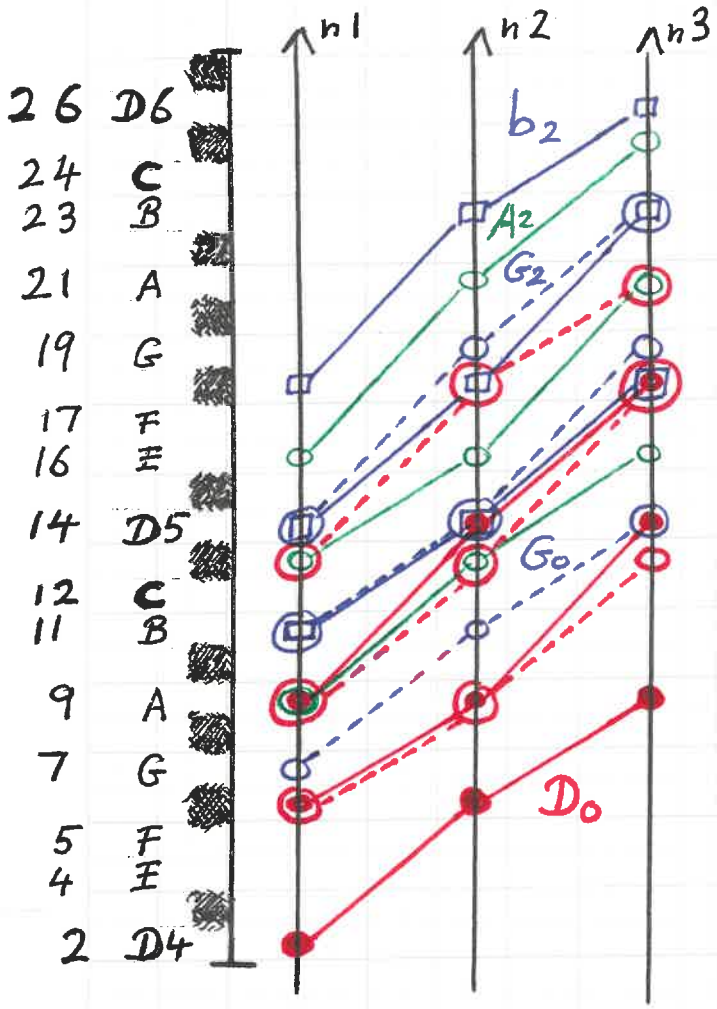
Based on this specific note-numbering, the lowest note (number) is D4 ($\hat{=}$ 2) and the highest note (number) is D6 ($\hat{=}$ 26) — i.e., these triads and their three positions are all viewed in the interval D4 – D6, defining 2 octaves.



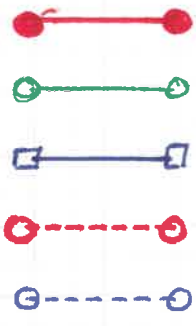
One can represent the three positions of the (sequence of) eight triads as "points" in a 3D space, using three integer coordinate axes n_1, n_2 and n_3 for the three stacked notes of a triad $(n_3, n_2, n_1)^T$, with n_1 being the lowest and n_3 being the highest note, see above figure. A point $\mathbb{P}$ is thus a "note-point" with integer coordinates n_1, n_2 and n_3: $\mathbb{P} = (n_1, n_2, n_3)^T$. Since the considered progression consists of eight triads, the progression can be thought of as a POLY-LINE CONNECTING EIGHT "NOTE-POINTS" (VERTICES) $\mathbb{P}_{\mathrm{I}}$, $\mathbb{P}_{\mathrm{V}}$, $\mathbb{P}_{\mathrm{vi}}$, $\mathbb{P}_{\mathrm{iii}}$, $\mathbb{P}_{\mathrm{IV}}$, $\mathbb{P}_{\mathrm{I}}$, $\mathbb{P}_{\mathrm{III}}$ and $\mathbb{P}_{\mathrm{V}}$. EACH OF THESE EIGHT POINTS IS A POINT THAT REPRESENTS ONE OF THREE TRIAD POSITIONS. Therefore, the number of possible 8-vertex polylines is $3^8 = 6561$. THE/AN OPTIMAL TRIAD PROGRESSION (POLY-LINE) MINIMIZES OVERALL NOTE-TO-NOTE MOVES.

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Parallel coordinates visualization of all triads.



D_0, D_1, D_2 I
A_0, A_1, A_2 V
b_0, b_1, b_2 vi
$f^\#_0, f^\#_1, f^\#_2$ iii
G_0, G_1, G_2 IV

The left figure uses a parallel-coordinates visualization for the three notes of all triads, considering all three positions for each triad. Geometrically, one must determine a set of eight polylines, satisfying the Pachelbel progression, that minimizes the value of the sum of all "up" and "down" movements of notes. One can employ a computational approach and calculate this sum of note movements for the 6561 possibilities to determine the/an optimal triad progression.

A triad progression "in the middle" of D4-D6 is

F#	E	F#	F#	G	F#	G	A
D	C#	D	C#	D	D	D	E
A	A	B	A	B	A	B	C#

(A4 → A5)

$\hat{=} D_2\text{-}A_0\text{-}b_0\text{-}f^\#_1\text{-}G_1\text{-}D_2\text{-}G_1\text{-}A_1$.

2	2	0	1	1	1	2	
1	1	1	1	0	0	2	
0	2	2	2	2	2	2	

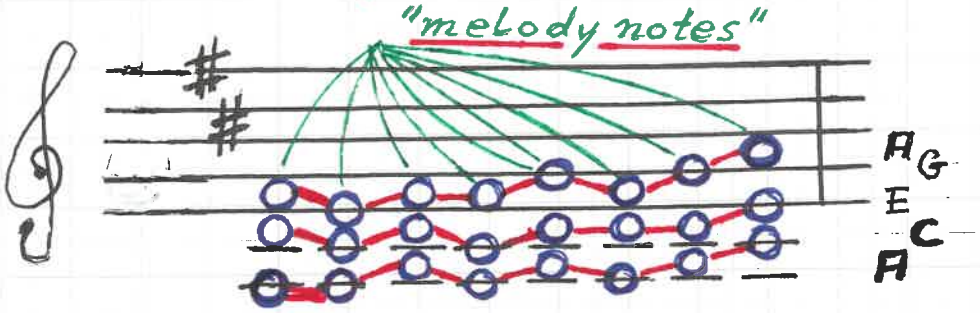
halftone ⇒ distances

Note-to-note halftone steps.

This progression only uses notes between A4 ($\hat{=}9$) and A5 ($\hat{=}21$). Note distances are ≤ 2.

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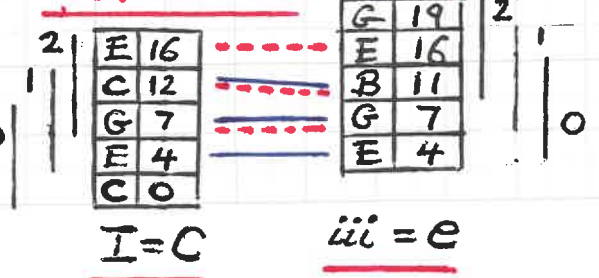
$I_2\ V_0\ vi_0\ iii_1\ IV_1\ I_2\ IV_1\ V_1$

$D_2\ A_0\ b_0\ f\#_1\ G_1\ D_2\ G_1\ A_1$

The left figure shows the chosen 8-triad progression that represents the Pachelbel progression. Again, this selection of 8 triads — with all triad notes in an octave-interval A-A — is just one of 6561 possibilities. It is not clear, i.e., it has not been computationally determined, whether this specific 8-triad progression in the key of D major is optimal in the sense of best-possible voice leading; nevertheless, one can "see" that all note-to-note movements when progressing from triad to triad are very small, i.e., less than 3 halftone steps (red line segments connecting notes).

• **Note.** The progression shown above involves a total of 27 halftone steps (using positive, absolute values of up- and down-movements only). Since this progression involves 21 (= 7×3) note-to-note movements, the average movement is $27/21 = 9/7 = 1\,2/7$ [steps].

Alternative for showing movement: 5-tuples



The left figure provides a visualization of C maj. and e min. triads, including all positions (0, 1, 2) and note numbers.

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{• Note. One can employ REAL-valued minimization methods to determine closest "position" of a triad (= the octave in which it is played) to a given "fixed" triad. This is a specific example:

$$C_0 = \begin{pmatrix} G \\ E \\ C \end{pmatrix} \hat{=} \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} ; \; G_0(j) = \begin{pmatrix} 14 \\ 11 \\ 7 \end{pmatrix} + \begin{pmatrix} 12j \\ 12j \\ 12j \end{pmatrix} \hat{=} \begin{pmatrix} D \\ B \\ G \end{pmatrix}$$

⇒ One must determine the j-value that "moves" G_0 as close as possible to C_0.

⇒ Use the "sum of squared note-to-note differences" and minimize it:

$$d^2 = (14+12j-7)^2 + (11+12j-4)^2 + (7+12j-0)^2$$
$$= 3(7+12j)^2 = 147 + 504j + 432j^2$$

$$\Rightarrow (d^2)' = 504 + 864j \doteq 0 \Rightarrow j = -7/12 .$$

One determines the (signed) integer J that is closest* to $j = -7/12$: $J = -1$. *floor: $\lfloor -7/12 \rfloor = -1$

⇒ The G_0 chord/triad closest to C_0 as given is the triad $G_0^{opt} = G_0(-1) =$

$$= \begin{pmatrix} 14 \\ 11 \\ 7 \end{pmatrix} + \begin{pmatrix} -12 \\ -12 \\ -12 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} \hat{=} \begin{pmatrix} D \\ B \\ G \end{pmatrix} = G_0 .$$

→ One can generalize this method for arbitrary triads:

$$ch1 = \begin{pmatrix} N_2 \\ N_1 \\ N_0 \end{pmatrix} ; \; ch2(j) = \begin{pmatrix} n_2+12j \\ n_1+12j \\ n_0+12j \end{pmatrix} \Rightarrow d^2 = \sum_{i=0}^{2} (n_i - N_i + 12j)^2 =$$

$$= \ldots = (\Delta_0^2 + \Delta_1^2 + \Delta_2^2) + 24(\Delta_0 + \Delta_1 + \Delta_2)j + 432j^2 , \; \Delta_i = n_i - N_i$$

$$\Rightarrow (d^2)' = 24(\Delta_0 + \Delta_1 + \Delta_2) + 864j \doteq 0 \Rightarrow j = -(\Delta_0 + \Delta_1 + \Delta_2)/36$$

By defining $\Delta = \Delta_0 + \Delta_1 + \Delta_2$, one obtains the optimal j-value as $j = -\Delta/36$. For the above example, this value is the value $-(7-0+11-4+14-7)/36 = -7/12$. }

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{• Note. Analogously, one can use REAL-value minimization to determine the closest "position" of three triads — the first triad being "fixed" and the second and third triads being "movable" by octaves. The three triads are:

$$ch1 = \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix};\quad ch2 = \begin{pmatrix} b_2 + 12j \\ b_1 + 12j \\ b_0 + 12j \end{pmatrix};\quad ch3 = \begin{pmatrix} c_2 + 12k \\ c_1 + 12k \\ c_0 + 12k \end{pmatrix}.$$

For example, one can add the "sums of squared note-to-note differences" for ch1 and ch2 and for ch2 and ch3 and minimize it:

$$d_{12}^2 = \sum_{i=0}^{3} (b_i - a_i + 12j)^2 = \sum_{i=0}^{3} (u_i + 12j)^2 =$$
$$= (u_0^2 + u_1^2 + u_2^2) + 24(u_0 + u_1 + u_2)j + 432j^2;$$
$$d_{23}^2 = \sum_{i=0}^{3} (c_i - b_i + 12(k-j))^2 = \sum_{i=0}^{3} (v_i + 12(k-j))^2 =$$
$$= (v_0^2 + v_1^2 + v_2^2) + 24(v_0 + v_1 + v_2)(k-j) + 432(k-j)^2$$
$$= (v_0^2 + v_1^2 + v_2^2) + 24(v_0 + v_1 + v_2)(k-j) + 432(k^2 - 2kj + j^2).$$

$$\Rightarrow d_{12}^2 = CON_{12} + 24uj + 432j^2;$$
$$d_{23}^2 = CON_{23} + 24v(k-j) + 432(k^2 - 2kj + j^2).$$

$$\Rightarrow D = d_{12}^2 + d_{23}^2 = CON + 24(u-v)j + 24vk + 864j^2 + 432k^2 - 864kj.$$

$$\Rightarrow \partial/\partial j\, D = 24(u-v) + 1728j - 864k \stackrel{!}{=} 0$$
$$\partial/\partial k\, D = 24v - 864j + 864k \stackrel{!}{=} 0$$

⇒ The following linear system must be solved:

$$72j - 36k = v - u \text{ and } 36j - 36k = v \Leftrightarrow$$
$$\Leftrightarrow 2j - k = (v-u)/36 \text{ and } j - k = v/36.$$
$$\Rightarrow j = -u/36;\quad k = -(u+v)/36.$$
$$\Rightarrow J = \text{ROUND}\lfloor j \rfloor;\quad K = \text{ROUND}\lfloor k \rfloor. \quad (\lfloor\,\rfloor = \text{floor})$$
$$\Rightarrow ch2_{opt} = \begin{pmatrix} b_2 + 12J \\ b_1 + 12J \\ b_0 + 12J \end{pmatrix};\quad ch3_{opt} = \begin{pmatrix} c_2 + K12 \\ c_1 + K12 \\ c_0 + K12 \end{pmatrix}.$$

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Example: We consider the 3-triad progression C_o (I), G_o (V) and F_o (IV) in the key of C:

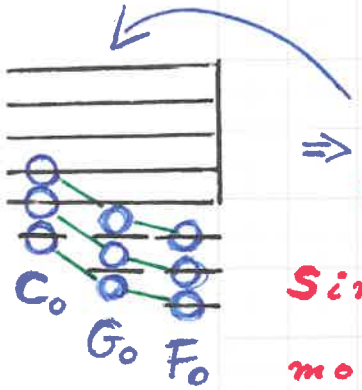
$$ch1=\begin{pmatrix}G\\E\\C\end{pmatrix}\hat{=}\begin{pmatrix}7\\4\\0\end{pmatrix};\quad ch2=\begin{pmatrix}D\\B\\G\end{pmatrix}\hat{=}\begin{pmatrix}14+12j\\11+12j\\7+12j\end{pmatrix};\quad ch3=\begin{pmatrix}C\\A\\F\end{pmatrix}\hat{=}\begin{pmatrix}0+12k\\-3+12k\\-7+12k\end{pmatrix}$$

$$\Rightarrow u=u_0+u_1+u_2=(7-0)+(11-4)+(14-7)=21$$

$$v=v_0+v_1+v_2=(-7-7)+(-3-11)+(0-14)=-42$$

$$\Rightarrow j=-u/36=-21/36=-7/12 \Rightarrow J=-1=\lfloor -7/12\rfloor$$

$$k=-(u+v)/36=21/36=7/12 \Rightarrow K=0=\lfloor 7/12\rfloor$$



$$\Rightarrow ch2_{opt}=\begin{pmatrix}D\\B\\G\end{pmatrix}\hat{=}\begin{pmatrix}14-12\\11-12\\7-12\end{pmatrix}=\begin{pmatrix}2\\-1\\-5\end{pmatrix};\quad ch3_{opt}=\begin{pmatrix}C\\A\\F\end{pmatrix}\hat{=}\begin{pmatrix}0+0\\-3+0\\-7+0\end{pmatrix}=\begin{pmatrix}0\\-3\\-7\end{pmatrix}$$

Since 4-chord / 4-triad progressions are the most commonly used progressions, we summarize the REAL-value minimization steps for 4 triads:

$$ch1=\begin{pmatrix}a_2\\a_1\\a_0\end{pmatrix};\quad ch2=\begin{pmatrix}b_2+12j\\b_1+12j\\b_0+12j\end{pmatrix};\quad ch3=\begin{pmatrix}c_2+12k\\c_1+12k\\c_0+12k\end{pmatrix};\quad ch4=\begin{pmatrix}d_2+12L\\d_1+12L\\d_0+12L\end{pmatrix}.$$

$$\Rightarrow d_{12}^2=(u_0^2+u_1^2+u_2^2)+24(u_0+u_1+u_2)j+432j^2.$$

$$d_{23}^2=(v_0^2+v_1^2+v_2^2)+24(v_0+v_1+v_2)(k-j)+432(k-j)^2.$$

$$d_{34}^2=(w_0^2+w_1^2+w_2^2)+24(w_0+w_1+w_2)(\ell-k)+432(\ell-k)^2.$$

$$\Rightarrow D=d_{12}^2+d_{23}^2+d_{34}^2=CON+24(uj+v(k-j)+w(\ell-k))$$
$$+432(j^2+(k-j)^2+(\ell-k)^2)$$
$$=CON+24((u-v)j+(v-w)k+w\ell)+432(2j^2+2k^2+\ell^2-2jk-2k\ell).$$

$$\Rightarrow \partial/\partial j\ D=24(u-v)+1728j-864k \doteq 0$$

$$\partial/\partial k\ D=24(v-w)-864j+1728k-864\ell \doteq 0$$

$$\partial/\partial \ell\ D=24w-864k+864\ell \doteq 0$$

$\Rightarrow$ The resulting linear system to be solved is:

$$2j-k=(v-u)/36 \text{ and } -j+2k-\ell=(w-v)/36 \text{ and } -k+j=-w/36$$

$$\Rightarrow j=-u/36;\ k=-(u+v)/36;\ \ell=-(u+v+w)/36$$

$$\Rightarrow J=\text{ROUND}[j];\ K=\text{ROUND}[k];\ L=\text{ROUND}[\ell].$$

• **Voice Leading & Optimization** 10/15/2024

Example: We consider the following progression:

$$C_0 \hat{=} \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix};\ G_0 \hat{=} \begin{pmatrix} 14+12J \\ 11+12J \\ 7+12J \end{pmatrix};\ F_0 \hat{=} \begin{pmatrix} 0+12K \\ -3+12K \\ -7+12K \end{pmatrix};\ a_0 \hat{=} \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} 12L \\ 12L \\ 12L \end{pmatrix} = \begin{pmatrix} d_2+12L \\ d_1+12L \\ d_0+12L \end{pmatrix}$$

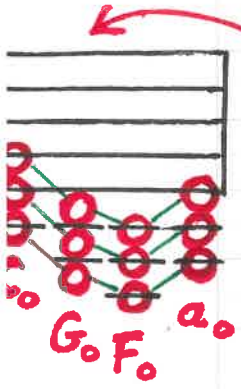
$\Rightarrow u = 21\ ;\ v = -42\ ;\ w = (-3+7)+(0+3)+(4-0) = 11$

$\Rightarrow j = -21/36 = -7/12\ ;\ k = -(21-42)/36 = 7/12\ ;$

$\ell = -(21-42+11)/36 = 5/18.$

$\Rightarrow J = -1\ ;\ K = 0\ ;\ L = 0\ \ (J=\lfloor -7/12 \rfloor;\ K=\lfloor 7/12 \rfloor;\ L=\lfloor 5/18 \rfloor)$

$\Rightarrow C_0 \hat{=} \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix};\ G^{\circ}_{opt} \hat{=} \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix};\ F^{\circ}_{opt} = \begin{pmatrix} 0 \\ -3 \\ -7 \end{pmatrix};\ a^{\circ}_{opt} \hat{=} \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}.$



This 4-triad progression of root-position triads C maj., G maj., F maj. and a min. is optimal: The total sum of up- and down-movements of corresponding notes of triad "neighbors" in the progression I-V-IV-vi is minimal (counting up- and down-movemnts of notes always as absolute – not signed – values). }

• **Note.** One can use this method for determining the "optimal" 8-triad progression, treated as an optimal voice leading problem, as follows: **1)** The first triad is given in a fixed location (in root position or in 1st or 2nd inversion). The other **7** triads are given in the order they should be played, considering all three possible positions for all **7** triads. Thus, the number of possible "triad picks" is $3^7 = 2187$. **2)** For each of these **2187** possibilities one determines the optimal "octave-placement." **3)** The best of these optimally placed triads wins.

• Voice Leading & Optimization

10/17/2024

⊛ Issue: Minimization of sum of all possible triad-to-triad distances vs. minimization of sum of triad-to-next-triad distances!

• Note. ⊛ "CORRECTION." When defining the distance measure $\mathcal{D}$ (to be minimized) for a 3-triad (4-triad) triad progression as discussed, one "must" CONSIDER ALL 3 (6) POSSIBLE TRIAD PAIR DISTANCES, i.e., d_{12}^2, d_{13}^2 and d_{23}^2 (d_{12}^2, d_{13}^2, d_{14}^2, d_{23}^2, d_{24}^2 and d_{34}^2). Thus, for the 3-triad progression minimization, we also can/"must" use $d_{13}^2 = \sum_{i=0}^{3}((c_i - a_i)+12k)^2 = \sum_{i=0}^{3}(w_i+12k)^2$.

The properly expanded expression for $\mathcal{D}$ becomes:

$$\mathcal{D}^2 = d_{12}^2 + d_{13}^2 + d_{23}^2 = CON + 24(u-v)j + 24(v+w)k + 864j + 864k^2 - 864jk$$

(since $d_{13}^2 = (w_0^2+w_1^2+w_2^2) + 24(w_0+w_1+w_2)k + 432k^2$ $= CON + 24wk + 432k^2$).

$$\Rightarrow \frac{\partial}{\partial j}\mathcal{D} = 24(u-v) + 1728j - 864k \stackrel{!}{=} 0$$
$$\frac{\partial}{\partial k}\mathcal{D} = 24(v+w) - 864j + 1728k \stackrel{!}{=} 0 .$$

⇒ The resulting linear system to be solved is:

$$(u-v) + 72j - 36k = 0 \;\wedge\; (v+w) - 36j + 72k = 0$$
$$\Leftrightarrow\; 2j - k = (v-u)/36 \;\wedge\; -j + 2k = (v+w)/36 .$$

⇒ The (real-valued) solution to this system is:

$$j = (-2u+v-w)/108 \,,\quad k = (-u-v-2w)/108 .$$

Example. Given the three triads

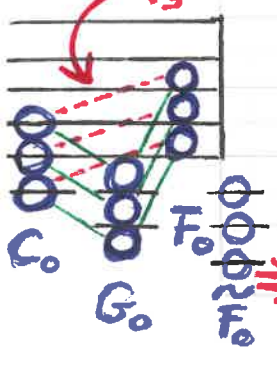
$$C_0 \hat{=} \begin{pmatrix}7\\4\\0\end{pmatrix};\quad G_0 \hat{=} \begin{pmatrix}14+12J\\11+12J\\7+12J\end{pmatrix};\quad F_0 \hat{=} \begin{pmatrix}0+12K\\-3+12K\\-7+12K\end{pmatrix}$$ (see p. 14/15),

the additional value $w = (-7-0)+(-3-4)+(0-7) = -21$ leads to the new values for j and k:

$$j = (-42-42+21)/108 = -\frac{7}{12};\quad k = (-21+42+42)/108 = \frac{7}{12}$$

$$\Rightarrow J = \text{ROUND}(-7/12) = -1;\quad K = \text{ROUND}(7/12) = 1 \Rightarrow G_0 \hat{=} \begin{pmatrix}2\\-1\\-5\end{pmatrix};\quad F_0 \hat{=} \begin{pmatrix}12\\9\\5\end{pmatrix}.$$

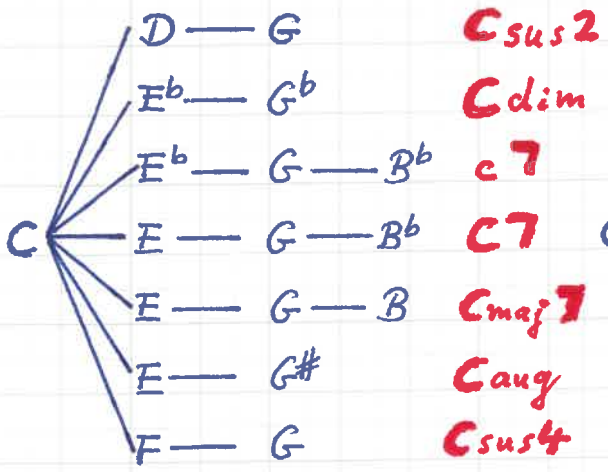
or: $J = \lfloor -7/12 \rfloor = -1$; $K = \lfloor 7/12 \rfloor = 0 \Rightarrow G_0 = (-5,-1,2)^T$; $\tilde{F}_0 = (-7,-3,0)^T$.

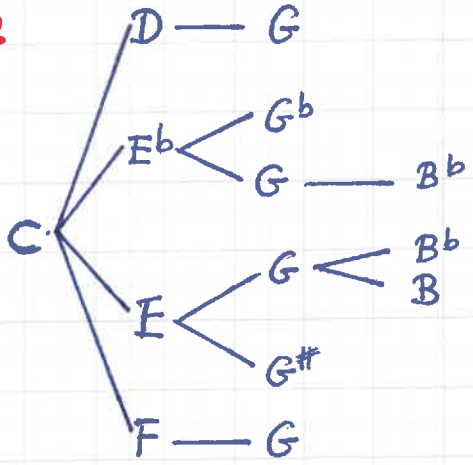


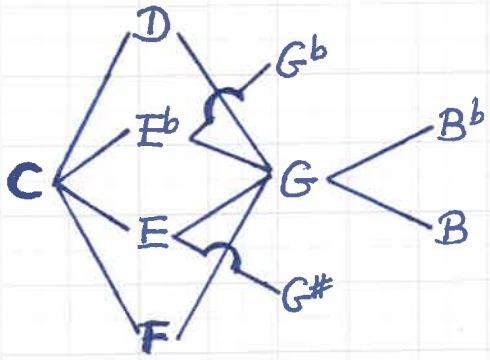
10/20/2024

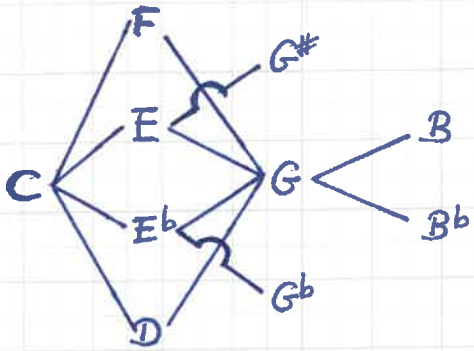
Effective Chord Representation

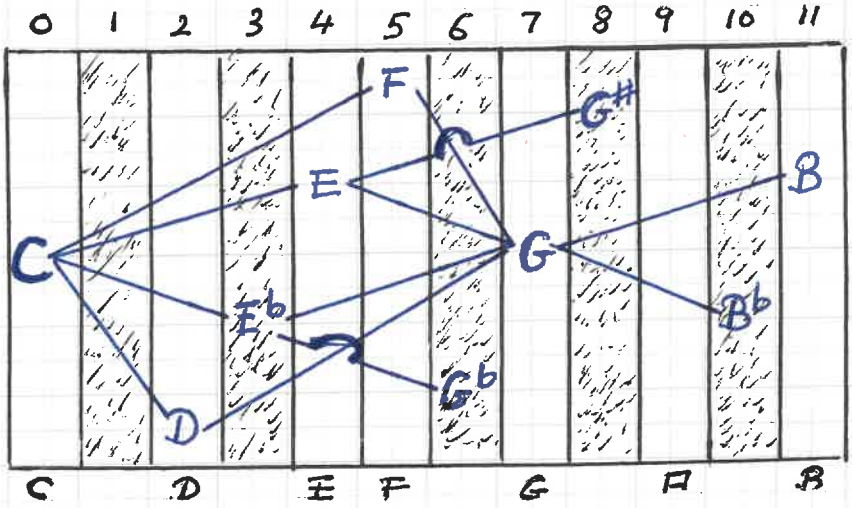
• Example. Representing chords with root note C







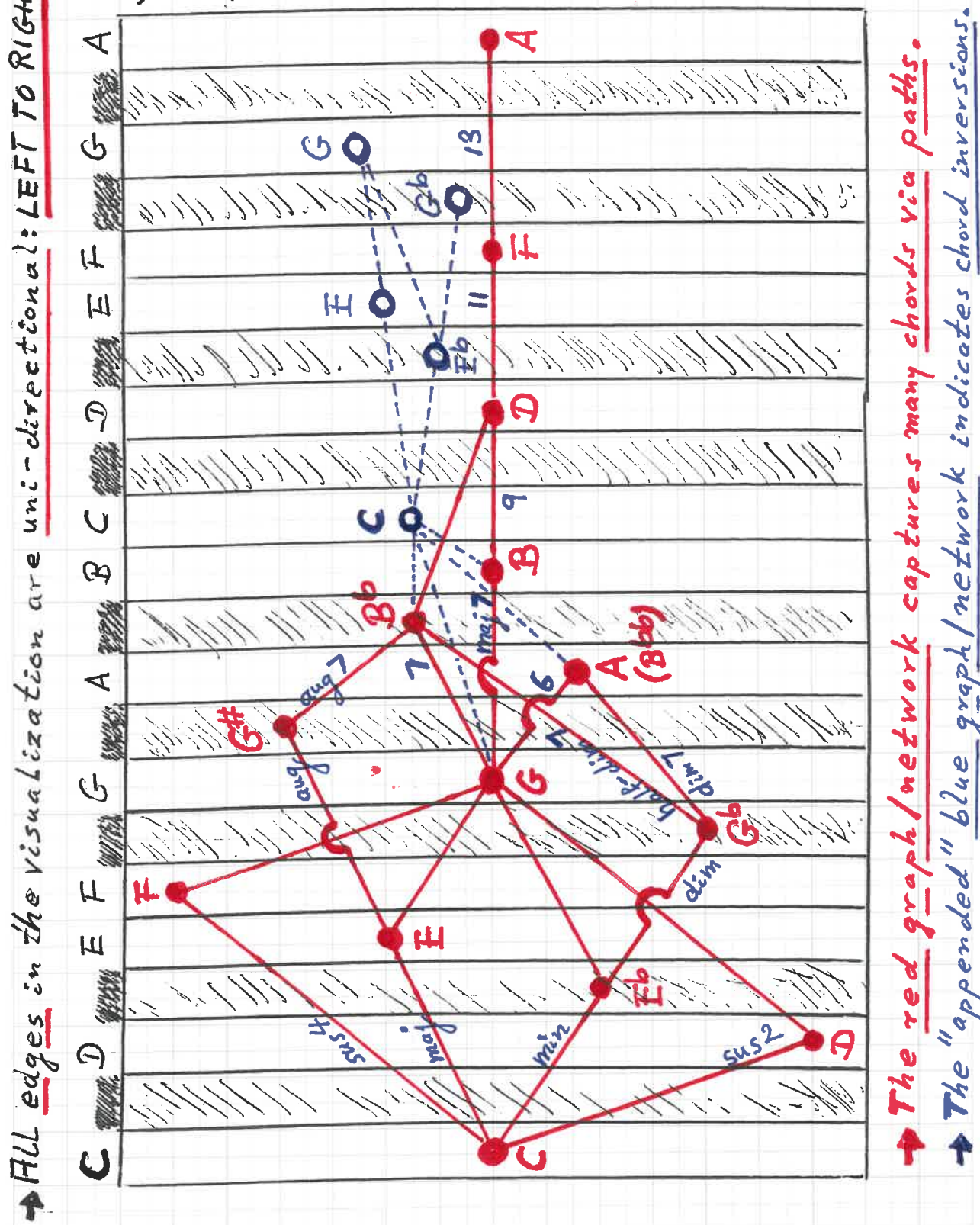




10/21/2024

"Effective Chord Representation

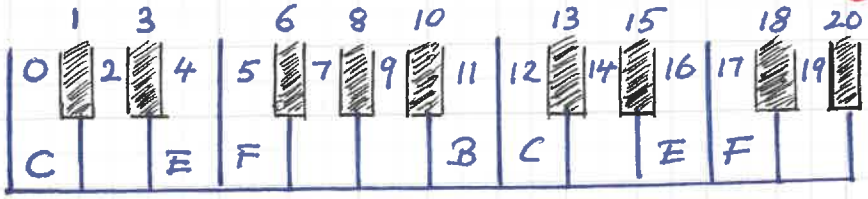
The visualization on this page captures some of the common chords with root note C : triads, 7th, 9th, 11th and 13th chords, major and minor.



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• Effective Chord Representation

10/23/2024



• **GOAL:** "Ideal and simple ways" to memorize "all chords."

• MAJOR Triads:

WWW ○○○
CEG, FAC, GBD

WBW ○●○
DF♯A, EG♯B, AC♯E

BWB ●○●
D♭FA♭, E♭GB♭, A♭CE♭

WBB ○●●
BD♯F♯

BWW ●○○
B♭DF

BBB ●●●
F♯A♯C♯ (=G♭B♭D♭)

(no WWB, BBW)

DUAL (WBW, BWB) — DUAL (WBB, BWW) — DUAL (WWW, BBB) — DUAL (all)

• MINOR Triads:

WWW ○○○
DFA, EGB, ACE

WBW ○●○
CE♭G, FA♭C, GB♭D

BWB ●○●
C♯EG♯, F♯AC♯, G♯BD♯

WWB ○○●
BDF♯

BBW ●●○
B♭D♭F

BBB ●●●
D♯F♯A♯ (=E♭G♭B♭)

(no WBB, BWW)

DUAL (WBW, BWB) — DUAL (WWB, BBW) — DUAL (all)

• DIMINISHED Triads:...

⊗ DIMINISHED Triads:

○○○ BDF
○○● DFA♭, EGB♭, ACE♭
○●○ FA♭C♭
●○○ C♯EG, F♯AC, G♯BD
○●● CE♭G♭, GB♭D♭
●●○ A♯C♯E, D♯F♯A

• Key-inherent Triads – Ex.: A major, F♯ minor

A B C♯ D E F♯ G♯ A B C♯ D

M: ○ ○ ● ○ ○ ● ● ○ — ACE
m: ○ ● ○ ○ ● ● ○ ○ — BDF♯
m: C♯EG♯
M: DF♯A
M: EG♯B
m: F♯AC♯
dim: G♯BD

3 MAJ (M), 3 MIN (m), 1 DIMINISHED (dim) chords.

F♯ G♯ A B C♯ D E F♯ G♯ A B

m: ● ● ○ ○ ● ○ ○ ● ● ○ ○ — F♯AC♯
dim: G♯BD
M: ACE
m: BDF♯
m: C♯EG♯
M: DF♯A
M: EG♯B

3 MAJ (M), 3 MIN (m), 1 DIMINISHED (dim) chords.

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Effective Chord Representation

Key-inherent Triads of Major and Minor Keys

	C	G	D	A	E	B	F#	C#
M	CEG	GBD	DF#A	AC#E	EG#B	BD#F#	F#A#C#	C#E#G#
m	DFA	ACE	EGB	BDF#	F#AC#	C#EG#	G#BD#	D#F#A#
m	EGB	BDF#	F#AC#	C#EG#	G#BD#	D#F#A#	A#C#E#	E#G#B#
M	FAC	CEG	GBD	DF#A	AC#E	EG#B	BD#F#	F#A#C#
M	GBD	DF#A	AC#E	EG#B	BD#F#	F#A#C#	C#E#G#	G#B#D#
m	ACE	EGB	BDF#	F#AC#	C#EG#	G#BD#	D#F#A#	A#C#E#
dim	BDF	F#AC	C#EG	G#BD	D#F#A	A#C#E	E#G#B	B#D#F#

F# = F##
C# = C##
G# = G##
D# = D##
A# = A##

E# = E## = F
B# = B## = C

Chords:

M - major

m - minor

dim - diminished

	F	Bb	Eb	Ab	Db	Gb	Cb
	FAC	BbDF	EbGB	AbCE	DbFA	GbBbD	CbEbG
	GBD	CEG	FAC	BbDF	EbGB	AbCE	DbFA
	ACE	DFA	GBD	CEG	FAC	BbDF	EbGB
	BbDF	EbGB	AbCE	DbFA	GbBbD	CbEbG	FAC
	CEG	FAC	BbDF	EbGB	AbCE	DbFA	GbBbD
	DFA	GBD	CEG	FAC	BbDF	EbGB	AbCE
	EGB	ACE	DFA	GBD	CEG	FAC	BbDF

Bb = B
Eb = E
Ab = A
Db = D
Gb = G

Cb = C = B
Fb = F = E

Colors:

blue-white keys

black-black keys

'-' sharp/flat

	a	e	b	f#	c#	g#	d#	a#
m	ACE	EGB	BDF#	F#AC#	C#EG#	G#BD#	D#F#A#	A#C#E#
dim	BDF	F#AC	C#EG	G#BD	D#F#A	A#C#E	E#G#B	B#D#F#
M	CEG	GBD	DF#A	AC#E	EG#B	BD#F#	F#A#C#	C#E#G#
m	DFA	ACE	EGB	BDF#	F#AC#	C#EG#	G#BD#	D#F#A#
m	EGB	BDF#	F#AC#	C#EG#	G#BD#	D#F#A#	A#C#E#	E#G#B#
M	FAC	CEG	GBD	DF#A	AC#E	EG#B	BD#F#	F#A#C#
M	GBD	DF#A	AC#E	EG#B	BD#F#	F#A#C#	C#E#G#	G#B#D#

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• Effective Chord Representation

10/24/2024

• The tables for the natural minor keys must be properly modified for the harmonic and melodic minor keys.

d	g	c	f	b^b	e^b	a^b
DFA	GBD	CEG	FAC	BDF	EGB	ACE
EGB	ACE	DFA	GBD	CEG	FAC	BDF
FAC	BDF	EGB	ACE	DFA	GBD	CEG
GBD	CEG	FAC	BDF	EGB	ACE	DFA
ACE	DFA	GBD	CEG	FAC	BDF	EGB
BDF	EGB	ACE	DFA	GBD	CEG	FAC
CEG	FAC	BDF	EGB	ACE	DFA	GBD

• Note. Only $2^3 = 8$ "key color combinations" are possible for a triad: WWW; WWB, WBW, BWW; WBB, BWB, BBW; or BBB. All triads listed in the triad tables above represent one of the 8 possibilities. CAN ONE USE THIS FACT FOR MEMORIZATION OF TRIAD SHAPES? DO "CLUSTERS OR GROUPS" OF SHAPES EXIST IN THE TABLES?

MEMORIZATION

• Note. KEY-INHERENT TRIADS

1) Given a specific key, major or minor, it has certain #s or bs.
2) The key has a root note and six subsequent notes.
3) For each of the seven notes, one defines a triad on top of it.
4) From an "alphabetical viewpoint," ONLY SEVEN BASIC TRIADS EXIST: CEG, DFA, EGB, FAC, GBD, ACE, BDF.
5) THE SPECIFIC KEY DEFINES WHICH OF THESE TRIADS IS THE FIRST AS KEY-INHERENT TRIAD (KEY=ROOT); THE #s AND b's OF THE KEY "MODIFY" THE BASIC TRIADS.

10/25/2024

• Effective Chord Representation

• Four-note Chords: 7th Chords

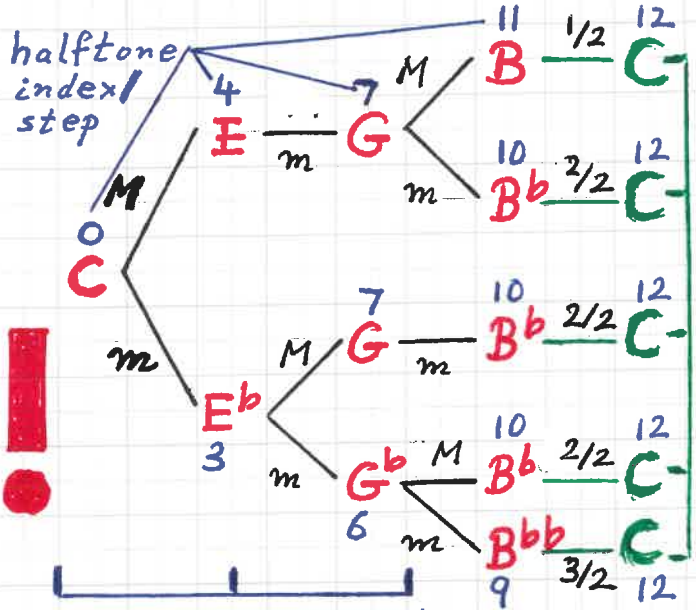
"Stacking major (M) and minor (m) 3rds":

[Example: C = root of 7th chord]

SAME 4 LETTERS: C E G B !

1) C E G B ≙ M m M (7th)
2) C E G B^b ≙ M m m (maj 7th)
3) C E^b G B^b ≙ m M m (min 7th)
4) C E^b G^b B^b ≙ m m M (half-diminished 7th)
5) C E^b G^b B^{bb} ≙ m m m ([fully] diminished 7th)

MEMORIZATION !



"Starting triads":

C E G (C maj.)
C E^b G (C min.)
C E^b G^b (C dimin.)

- The last possible notes of a 7th chord are: B, B^b, B^{bb} (≙A).
- These three notes are 1/2, 2/2, 3/2 HALF-TONES "BELOW C," C being one octave above the root C.
- Note. 1) Given a specific root note and type of 7th chord, determine and play the "starting triad."

2) Depending on the type of 7th chord, play a B, B^b or B^{bb} (≙A) being 1/2, 2/2 or 3/2 steps below the C that is one octave above the chord's root C.

- Note. For 4-note chords $2^4 = 16$ "key color combinations" are possible, from WWWW to BBBB. For example, WWWW, WWWB, WBWB, WBBB, WBBW apply to the C chords.

10/25/2024

- **Effective Chord Representation**

- **But:** Due to the large number of 7th chords when considering all possible root notes/keys and 16 possible white-and-black "key color combinations", it seems that it is not a viable strategy to memorize 7th chord shapes by considering the 16 possibilities.

- **Table of five 7th chords for 12 root notes:**

Key/root	MmM	Mmm	mMm	mmM	mmm
C	CEGB	CEG°B	C°EG°B	C°E°G°B	C°E°G°°B
G	GBDF°	GBDF	G°BDF	G°B°DF	G°B°D°F
D	DF°AC°	DF°AC	DF♯AC	DF°AC	DF°A°C
A	AC°EG°	AC°EG	ACEG	AC°EG	AC°E°G
E	EG°BD°	EG°BD	EGBD	EG°BD	EG°B°D
B	BD°F°A°	BD°F°A	BDF°A	BDF♯A	BDF°A
F♯	F°A°C°E°	F°A°C°E	F°AC°E	F°ACE	F°AC°E
C♯	C°E°G°B°	C°E°G°B	C°EG°B	C°EGB	C°EG°B
F	FACE	FAC°E	F°AC°E	F°A°C°E	F°A°C°°E
B♭	°BDFA	°BDF°A	°B°DF°A	°B°D°F°A	°B°D°F°°A
E♭	°EG°BD	°EG°B°D	°E°G°B°D	°E°G°°B°D	°E°G°B°°D
A♭	°AC°EG	°AC°E°G	°A°C°E°G	°A°C°°E°G	°A°C°°E°°G
D♭	°DF°AC	°DF°A°C	°D°F°A°C	°D°F°°A°C	°D°F°°A°°C
G♭	°G°B°DF	°G°B°D°F	°G°°B°D°F	°G°°B°°D°F	°G°°B°D°°F
C♭	°C°E°G°B	°C°E°G°°B	°C°°E°G°°B	°C°°E°°G°°B	°C°°E°°G°°°B

BLUE: white key

BLACK black key

EQUIVALENT

12×5 = 60 7th chords; 3 inversions per chord ⇓ 240 patterns

Notation: F° = F♯, °B = B♭, °°B = B♭♭, °°°B = B♭♭♭ etc.

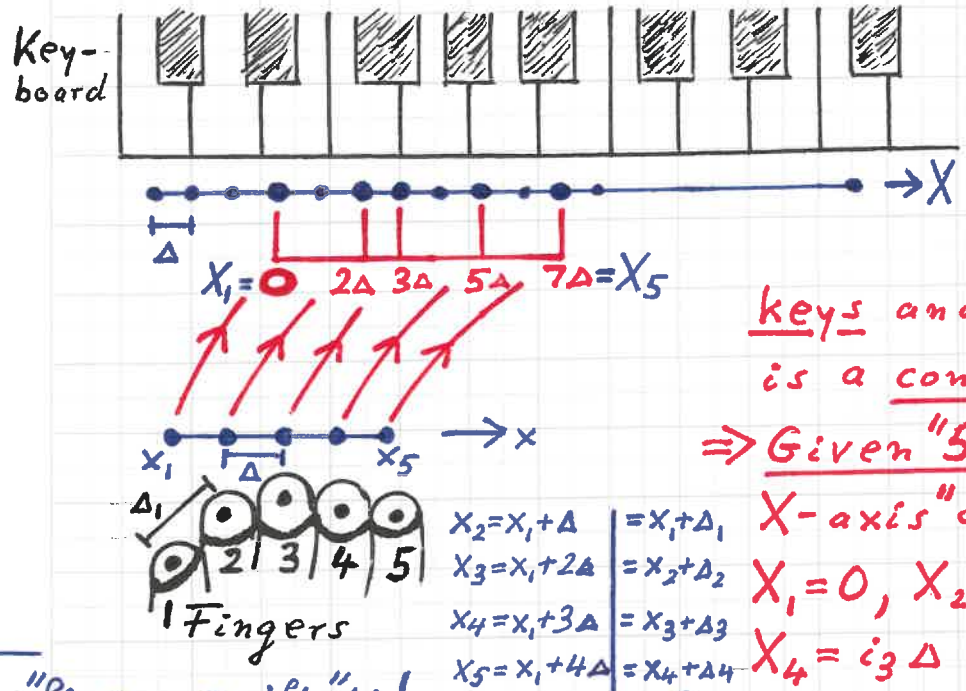
E♯ ≙ F, B♯ ≙ C, C♭ ≙ B, F♭ ≙ E.

MmM ≙ M3rd + m3rd + M3rd etc.

10/26/2024

• Optimization and Fingering

• Given a chord, a note sequence or a "block of notes," what is a good, near-optimal fingering?



SIMPLIFICATION: The physical "distance" between halftone keys and between fingers is a constant: Δ.

⇒ Given "5 notes on the X-axis" at X-positions $X_1=0$, $X_2=i_1\Delta_1$, $X_3=i_2\Delta_1$, $X_4=i_3\Delta$ and $X_5=i_4\Delta$) what locations are "ideal" for the fingers? The unknown finger locations on the x-axis — x_1, $x_1+\Delta_1$, $x_1+2\Delta_1$, $x_1+3\Delta$, $x_1+4\Delta$ — must be chosen to MINIMIZE THE SUM OF THE SQUARED DISTANCES BETWEEN X_i and x_i, $i=1\ldots5$:

$$X_2=X_1+\Delta_1 \quad | \quad =X_1+\Delta_1$$
$$X_3=X_1+2\Delta \quad | \quad =X_2+\Delta_2$$
$$X_4=X_1+3\Delta \quad | \quad =X_3+\Delta_3$$
$$X_5=X_1+4\Delta \quad | \quad =X_4+\Delta_4$$

"finger-specific" Δ_i!

$$X_2=X_1+\Delta_1$$
$$X_3=X_1+\Delta_1+\Delta_2$$
$$X_4=X_1+\Delta_1+\Delta_2+\Delta_3$$
$$X_5=X_1+\Delta_1+\Delta_2+\Delta_3+\Delta_4$$

Δ_i: "anatomy of fingers"

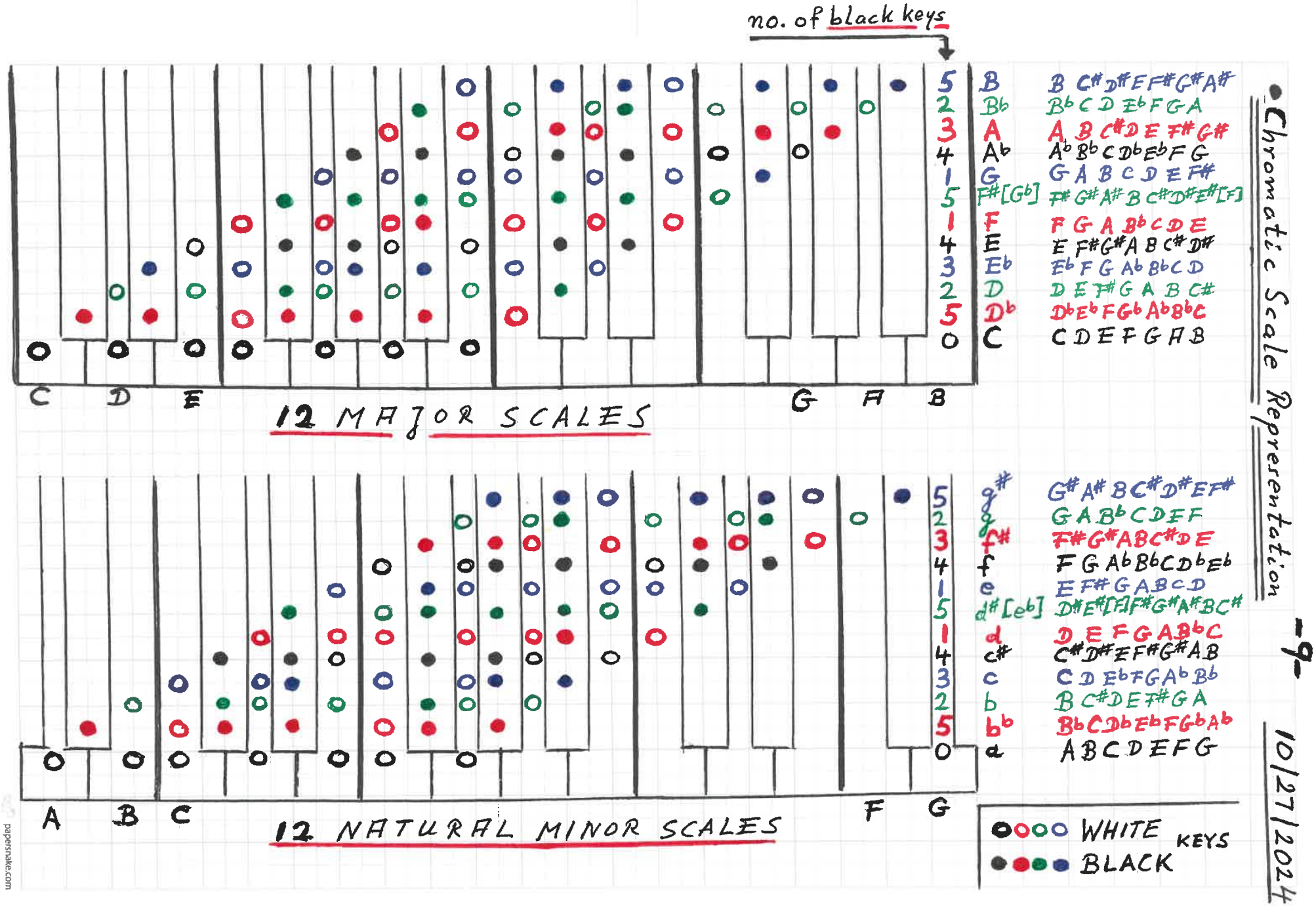
$$D=\sum_{i=1}^{5}(d_i)^2=(x_1-0)^2+(x_1+\Delta-i_1\Delta)^2+(x_1+2\Delta-i_2\Delta)^2$$
$$+(x_1+3\Delta-i_3\Delta)^2+(x_1+4\Delta-i_4\Delta)^2$$
$$=5x_1^2+2\Delta x_1\left(10-(i_1+i_2+i_3+i_4)\right)+\text{CONST}$$
$$=5x_1^2-2\Delta x_1\left(i_1+i_2+i_3+i_4-10\right)+\text{CONST}$$

$$\Rightarrow d/dx_1\, D=10x_1-2\Delta\left(i_1+i_2+i_3+i_4-10\right)\overset{!}{=}0$$
$$\Rightarrow x_1=\Delta/5\left(i_1+i_2+i_3+i_4-10\right)$$

Example: $X_1=0$, $X_2=1\Delta$, $X_3=2\Delta$, $X_4=3\Delta$, $X_5=4\Delta$

$$\Rightarrow i_1+i_2+i_3+i_4=1+2+3+4=10$$
$$\Rightarrow x_1=\Delta/5\,(10-10)=\Delta/5\cdot0=0=X_1\ \checkmark$$

Chromatic Scale Representation



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• Key-inherent Chords & Inversions

10/29/2024

➡ How to memorize these chords?

C	D	E	F	G	A	B	C	D	E	F	G	A	B	C	D	E	F
C		E		G		7th B	C		E		G						
I						root		1st		2nd		3rd	inversions				
	D		F		A		C	D		F		A					
ii		E		G		B		D	E		G		B				
iii		F		A		C			E	F		A		C			
IV		G		B		D				F	G		B		D		
V		A		C		E					G	A		C		E	
vi		B		D		F						A	B		D		F
vii°																	

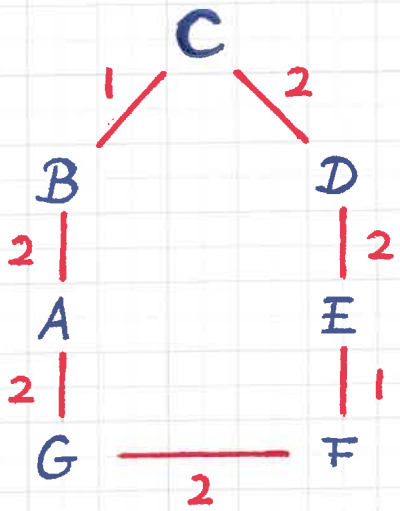
- This example shows all C-major-inherent triads and 7th chords and all inversions!
- One only needs to memorize these SEVEN letter sequences — since the same letter sequences apply to all major and minor keys!
- When a specific key signature is given — i.e., C, G, D, ..., F, B♭, ..., a, ... — one knows that one must apply '#'s and 'b's to certain letters / notes; thus, the seven letters remain the same and are merely modified by the '#' and 'b' symbols!
- Analogously, one can also think of the symbols '#' and 'b' as simple "modifiers of white keys": A '#' implies "playing to the right," a 'b' "playing to the left"!

• Church Modes (Modi) – Kirchentonarten

10/31/2024

• The seven church modes "on the key of C" are:

I	IONIAN = C major	C ^{2} D ^{2} E ^{1} F ^{2} G ^{2} A ^{2} B ^{1} C
ii	DORIAN	D ^{2} E ^{1} F ^{2} G ^{2} A ^{2} B ^{1} C ^{2} D
iii	PHRYGIAN	E ^{1} F ^{2} G ^{2} A ^{2} B ^{1} C ^{2} D ^{2} E
IV	LYDIAN	F ^{2} G ^{2} A ^{2} B ^{1} C ^{2} D ^{2} E ^{1} F
V	MIXOLYDIAN	G ^{2} A ^{2} B ^{1} C ^{2} D ^{2} E ^{1} F ^{2} G
vi	AEOLIAN = a minor	A ^{2} B ^{1} C ^{2} D ^{2} E ^{1} F ^{2} G ^{2} A
vii°	LOCRIAN (HYPOPHRYGIAN)	B ^{1} C ^{2} D ^{2} E ^{1} F ^{2} G ^{2} A ^{2} B



• The seven modes all have 5 whole-tones ('2') and 2 half-tones ('1') between the seven pairs of subsequent notes.

• The sequence of notes CDEFGAB is cyclic, and the half-tone steps are always the steps E-F and B-C.

• The note used as first note determines the specific mode and "nature" of an octave.

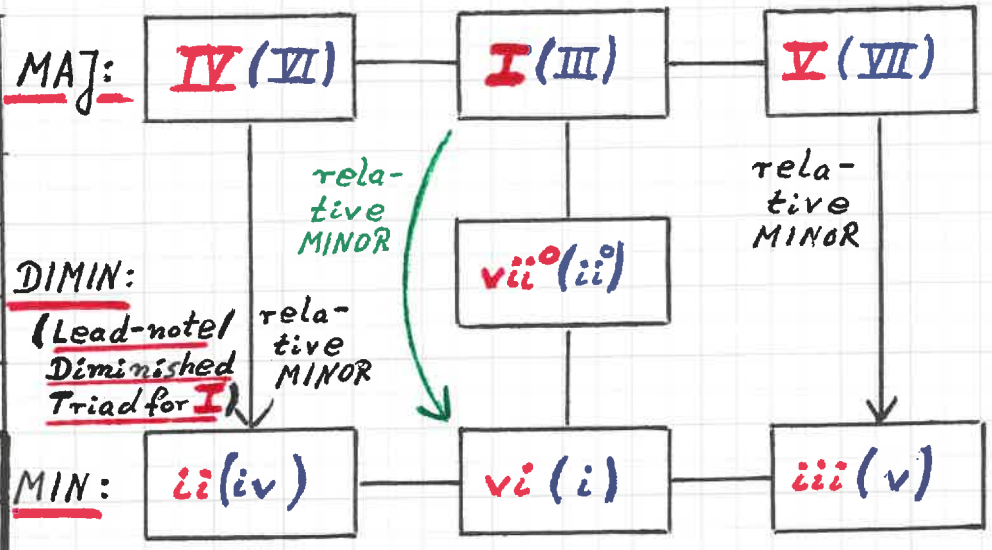
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• The Circle of Fifths and its Relationship with Key-inherent Triads

MAJOR	LEAD-NOTE/ DIM. CHORD (TRIAD)	MINOR
C#	$b^{\#\circ}$	$a^{\#}$
F#	$e^{\#\circ}$	$d^{\#}$
B	$a^{\#\circ}$	$g^{\#}$
E	$d^{\#\circ}$	$c^{\#}$
A	$g^{\#\circ}$	$f^{\#}$
D	$c^{\#\circ}$	b
G	$f^{\#\circ}$	e
C	b°	a
F	e°	d
Bb	a°	g
Eb	d°	c
Ab	g°	f
Db	c°	bb
Gb	f°	eb
Cb	bb°	ab

COMPACT CIRCLE OF 5THS

• Every "GROUP" in the circle can be viewed as a group of seven key-inherent triads — representing both MAJOR AND MINOR keys/triads:



Use case: Given a key for I, the resulting sequence of key-inherent triads consists of the triads I maj, ii min, ..., vi min, vii° dimin.

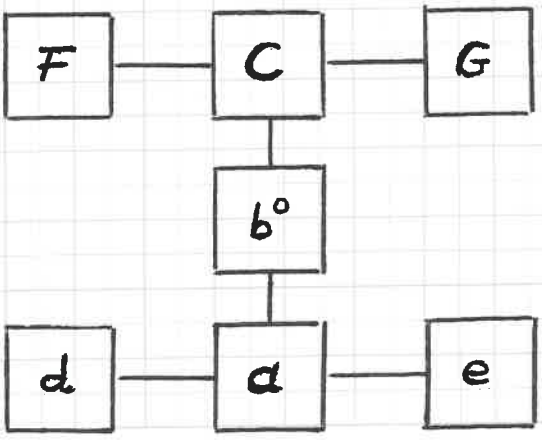
• The seven Roman numerals represent the seven notes C, D, E, F, G, A, B.

• I, IV, V (III, VI, VII): MAJOR key-inherent triad implied

• ii, iii, vi, vii° (i, ii°, iv, v): MINOR key-inherent triad (and one DIMINISHED triad (or [or])) implied.

• The ORDER defined by the seven Roman numerals establishes the key-inherent order of key-inherent triads, i.e., from I (i) to vii° (VII).

• For example, the GROUP FOR I ≙ C leads to:

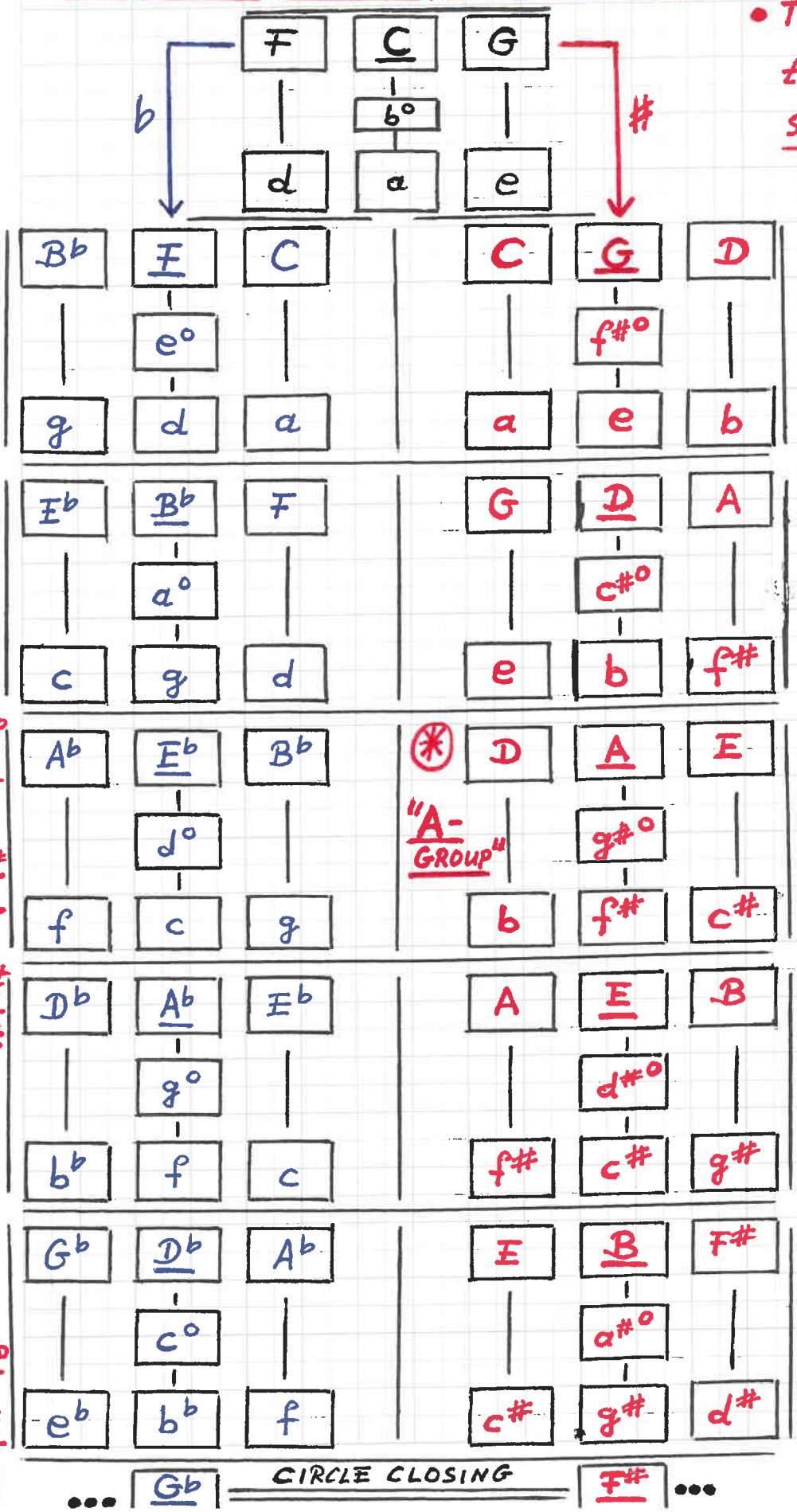


• The two implied triad sequences are:
CEG, DFA, EGB, FAC, GBD, ACE, BDF;
ACE, BDF, CEG, DFA, EGB, FAC, GBD.

• C maj. and its relative minor, a min., have no sharps or flats. No sharps or flats occur in the seven triads.

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• The Circle of Fifths and its Relationship with Key-inherent Triads



• This figure shows the "GROUPS OF SEVEN" for the keys C, G, D, A, E, B and F, B♭, E♭, A♭, D♭.

• The figure uses the order of the circle of fifths.

• The four groups not shown are these:

B	F♯	C♯
	e♯°	
g♯	d♯	a♯

F♯	C♯	G♯
	b♯°	
d♯	a♯	e♯

C♭	G♭	D♭
	f°	
a♭	e♭	b♭

F♭	C♭	G♭
	b♭°	
d♭	a♭	e♭

• The circle is closing at the bottom, since F♯ ≙ G♭.

✱ USE:

1) The group for A maj. has notes A, B, C♯, D, E, F♯, G♯.

2) The resulting A-inherent triad sequence is:
A C♯ E
B D F♯
C♯ E G♯
D F♯ A
E G♯ B
F♯ A C♯
G♯ B D

THE A-GROUP LISTS THE NOTES OF A-INHERENT TRIADS!

✱ "A-GROUP"

• THE KEY-INHERENT TRIADS ARE DEFINED BY THE GROUPS OF 7.

→ Circle of 5ths | Group of 7 for C | TP, DP, SP: Tonic, Dominant, Subdominant | PARALLEL (min.)

Major	C^{b}	G^{b}	D^{b}	A^{b}	E^{b}	B^{b}	F	C	G	D	A	E	B	$F^{\#}$	$C^{\#}$	maj. chords
1 Lead-note	$b^{b\circ}$	f°	c°	g°	d°	a°	e°	b°	$f^{\#\circ}$	$c^{\#\circ}$	$g^{\#\circ}$	$d^{\#\circ}$	$a^{\#\circ}$	$e^{\#\circ}$	$b^{\#\circ}$	dimin. chords
2 Minor	a^{b}	e^{b}	b^{b}	f	c	g	d	a (TP)	e	b	$f^{\#}$	$c^{\#}$	$g^{\#}$	$d^{\#}$	$a^{\#}$	minor chords
2	G^{b}	D^{b}	A^{b}	E^{b}	B^{b}	F	C	G	D	A	E	B	$F^{\#}$	$C^{\#}$	$G^{\#}$	DOMINANT
2	F^{b}	C^{b}	G^{b}	D^{b}	A^{b}	E^{b}	B^{b}	F	C	G	D	A	E	B	$F^{\#}$	SUBDOMINANT
1	E^{b}	B^{b}	F	C	G	D	A	E	B	$F^{\#}$	$C^{\#}$	$G^{\#}$	$D^{\#}$	$A^{\#}$	$E^{\#}$	DP
2	D^{b}	A^{b}	E^{b}	B^{b}	F	C	G	D	A	E	B	$F^{\#}$	$C^{\#}$	$G^{\#}$	$D^{\#}$	SP
2	C^{b}	G^{b}	D^{b}	A^{b}	E^{b}	B^{b}	F	C	G	D	A	E	B	$F^{\#}$	$C^{\#}$	TONIC

↑ Notes of keys

No. of half-tone steps between notes (MUSICIAN)

- Major row: Major triads to be constructed on listed notes
- Lead-note row: Diminished triads to be constructed on listed notes
- Minor row: Minor triads to be constructed on listed notes
- Example, group for C: I, ii, iii, IV, V, vi, vii° = CEG, DFA, EGB, FAC, GBD, ACE, BDF
  (C maj. and a min. keys) i, ii°, III, iv, v, VI, VII = ACE, BDF, CEG, DFA, EGB, FAC, GBD

• The Circle of Fifths and its Relationship with Key-inherent Triads

11/2/2024

• CHORD RECOGNITION FROM NOTES

11/11/2024

"How to recognize the chords that (possibly) define the underlying harmonic structure — by considering the given notes?"

• Here: 1, 2, 3, 4, 5, 6, 7 represent the "seven ordered notes of a key, e.g., {1,2,3,4,5,6,7} = {C, D, E, F, G, A, B}.

• We write the seven key-inherent triads and their inversions as follows (maj. and min. triads):

I: 1, 3, 5 ; 3, 5, 8(1) ; 5, 8(1), 10(3)

II: 2, 4, 6 ; 4, 6, 9(2) ; 6, 9(2), 11(4)

III: 3, 5, 7 ; 5, 7, 10(3) ; 7, 10(3), 12(5)

IV: 4, 6, 8(1) ; 6, 8(1), 11(4) ; 8(1), 11(4), 13(6)

V: 5, 7, 9(2) ; 7, 9(2), 12(5) ; 9(2), 12(5), 14(7)

VI: 6, 8(1), 10(3) ; 8(1), 10(3), 13(6) ; 10(3), 13(6), 15(8)=1

VII: 7, 9(2), 11(4) ; 9(2), 11(4), 14(7) ; 11(4), 14(7), 16(9)=2

Ex.: 1,3,5 = CEG ; 2,4,6 = DFA ; 3,5,7 = EGB ; ... ; 7,9,11 = BDF.

• We write the seven key-inherent 7th chords and their inversions as follows (maj. and min 7th chords):

I: 1, 3, 5, 7 ; 3, 5, 7, 8(1) ;
5, 7, 8(1), 10(3) ; 7, 8(1), 10(3), 12(5) ;

II: 2, 4, 6, 8(1) ; 4, 6, 8(1), 9(2) ;
6, 8(1), 9(2), 11(4) ; 8(1), 9(2), 11(4), 13(6) ;

III: 3, 5, 7, 9(2) ; 5, 7, 9(2), 10(3) ;
7, 9(2), 10(3), 12(5) ; 9(2), 10(3), 12(5), 14(7) ;

IV: 4, 6, 8(1), 10(3) ; 6, 8(1), 10(3), 11(4) ;
8(1), 10(3), 11(4), 13(6) ; 10(3), 11(4), 13(6), 15(1) ;

V: 5, 7, 9(2), 11(4) ; 7, 9(2), 11(4), 12(5) ;
9(2), 11(4), 12(5), 14(7) ; 11(4), 12(5), 14(7), 16(2)

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• CHORD RECOGNITION FROM NOTES

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VI: 6, 8(1), 10(3), 12(5); 8(1), 10(3), 12(5), 13(6);
10(3), 12(5), 13(6), 15(1); 12(5), 13(6), 15(1), 17(3)

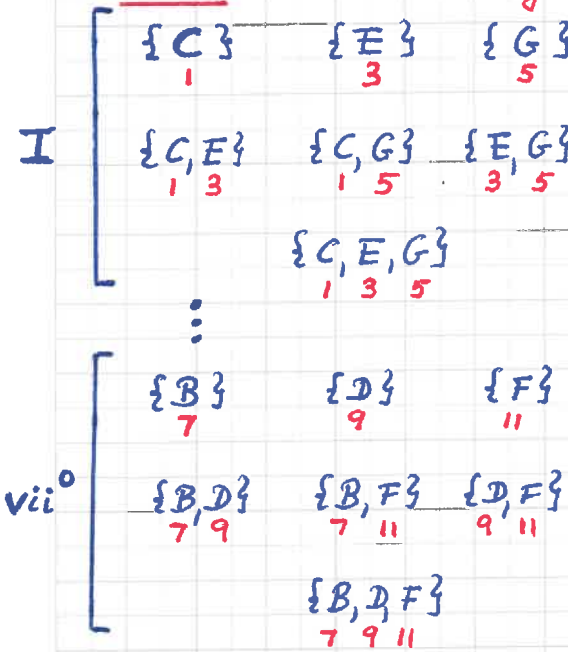
VII: 7, 9(2), 11(4), 13(6); 9(2), 11(4), 13(6), 14(7);
11(4), 13(6), 14(7), 16(2); 13(6), 14(7), 16(2), 18(4).

Ex.: 1, 3, 5, 7 = CEGB; 2, 4, 6, 8(1) = DFAC;
3, 5, 7, 9(2) = EGBD; ...; 7, 9(2), 11(4), 13(6) = BDFA.

• Note. MODULO-ARITHMETIC is used for notes / numbering, i.e., when considering the key of C:
C ≙ 1, 8, 15, 22; D ≙ 2, 9, 16, 23; B ≙ 7, 14, 21, 28.

⇒ ONE CAN "GROUP A SEQUENCE OF NOTES TOGETHER AND ORDER THEM" (FROM LOW TO HIGH) TO ESTABLISH "NOTE SETS" THAT DEFINE POSSIBLE "CHORD SUBSETS", I.E., POTENTIALLY UNDERLYING CHORDS!

Ex.: Considering the key of C major, the seven key-inherent triads CEG, ..., BDF can occur in the form of "incomplete triads," i.e., as "note subsets in a score, see left figure. Mathematically, a set with three elements (notes) has one subset with three notes and three subsets with two notes and one note, respectively. FOR EXAMPLE, THE NOTE SUBSET {C, E} CAN BE A SUBSET OF {C, E, G} OR {A, C, E}.



The seven subsets of possible note subsets defining the seven triads I, ii, iii, IV, V, vi and vii° in C.

• CHORD RECOGNITION FROM NOTES

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⇒ One can systematically determine all triads and/or 7th chords that could potentially reflect an underlying harmonic chord structure/harmony. Thus, one must "map" all selected/extracted note subsets (with ordered notes) to all viable triads and/or 7th chords of which they are (or could be) a part.

Note. By using the concept of "note sets" with the implicit order of notes from Low to high within the sequence C(1)D(2)E(3)F(4)G(5)A(6)B(7), for example, a triad or 7th chord note set defines the chords in root position and their inversions: {C, E, G} defines the triads CEG, EGC and GCE; {C, E, G, B} defines the 7th chords CEGB, EGBC, GBCE and BCEG.

TRIADS

I	II	III	IV	V	VI	VII
CEG	DFA	EGB	FAC	GBD	ACE	BDF
CE, CG, EG	DF, DA, FA	EG, EB, GB	FA, FC, AC	GB, GD, BD	AC, AE, CE	BD, BF, DF
C, E, G	D, F, A	E, G, B	F, A, C	G, B, D	A, C, E	B, D, F

⇒ Each note (C, D, E, F, G, A, B) is part of THREE triads.

The 3rd's (CE, EG, DF, FA, EG, GB, FA, AC, GB, BD, AC, CE, BD, DF) are part of TWO triads.

The 5th's (CG, DA, EB, FC, GD, AE, BF) are part of ONE triad.

7th CHORDS

I	...	VII
CEGB		BDFA
CEG, CEB, CGB, EGB		BDF, BDA, BFA, DFA
CE, CG, CB, EG, EB, GB		BD, BF, BA, DF, DA, FA
C, E, G, B		B, D, F, A

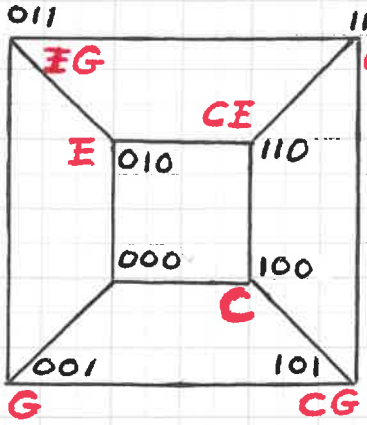
7th chords and their "subset chords" (2-note, 3-note and 1-note chords)

⇐ Notes occur 4 times, 3rd's occur 3 times, 5th's occur 2 times and 7th's occur once in the seven 7th chords. (Notes = C, D, ...; 3rd's = CE, EG, ...; 5th's = CG, EB, ...; 7th's = CB, DC, ...)

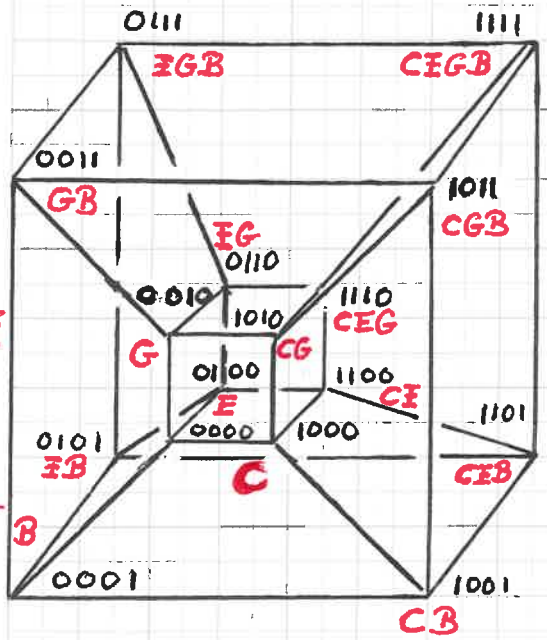
• CHORD RECOGNITION FROM NOTES

11/12/2024

$2^3 = 8$ triad possibilities for $b_1 b_3 b_5$



$2^4 = 16$ 7th chord possibilities for $b_1 b_3 b_5 b_7$



In order to determine a (the) chord(s) possibly implied and played by a sequence of notes, one can think of a bit sequence that determines which notes occur in the considered sequence. Since the seven "base notes" are C, D, E, F, G, A and B, the bit sequence $b_1 b_2 b_3 b_4 b_5 b_6 b_7$ can indicate the notes that exist in a sequence. For example, 1010101 means that the notes C, E, G and B are represented (each of these four notes occurring at least once in the considered sequence). Further, the 3-bit (4-bit) sequence $b_1 b_3 b_5$ ($b_1 b_3 b_5 b_7$) can be used to define the observed note subset that possibly identifies a note subset of the triad CEG (top figure) (of the 7th chord CEGB (bottom figure)). Here, the bit sequence $b_1 b_2 b_3 b_4 b_5 b_6 b_7$ corresponds to the associated key sequence C, D, E, F, G, A, B ($\hat{=}$ I). Of course, the sequence can also be used to represent the other possible key sequences, i.e., D, E, F, G, A, B, C, ..., and B, C, D, E, F, G, A ($\hat{=}$ II to VII). Again, 3-bit (4-bit) bit sequences can be used for triad (7th chord) note subsets.

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• CHORD RECOGNITION FROM NOTES

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Here, it is assumed that the following note-index correspondences apply: C ≙ 1, D ≙ 2, E ≙ 3, F ≙ 4, G ≙ 5, A ≙ 6 and B ≙ 7. For example, one can identify a 4-note 7th chord via 4-bit bit sequences — "over C, D, E, F, G, A and B." Specifically, the seven 7th chords and associated bit sequences are:

CEGB	≙ $b_1 b_3 b_5 b_7$	CEG	≙ $b_1 b_3 b_5$
DFAC ≙ CDFA	≙ $b_1 b_2 b_4 b_6$	DFA	≙ $b_2 b_4 b_6$
EGBD ≙ DEGB	≙ $b_2 b_3 b_5 b_7$	EGB	≙ $b_3 b_5 b_7$
FACE ≙ CEFA	≙ $b_1 b_3 b_4 b_6$	FAC ≙ CFA	≙ $b_1 b_4 b_6$
GBDF ≙ DFGB	≙ $b_2 b_4 b_5 b_7$	GBD ≙ DGB	≙ $b_2 b_5 b_7$
ACEG ≙ CEGA	≙ $b_1 b_3 b_5 b_6$	ACE ≙ CEA	≙ $b_1 b_3 b_6$
BDFA ≙ DFAB	≙ $b_2 b_4 b_6 b_7$	BDF ≙ DFB	≙ $b_2 b_4 b_7$

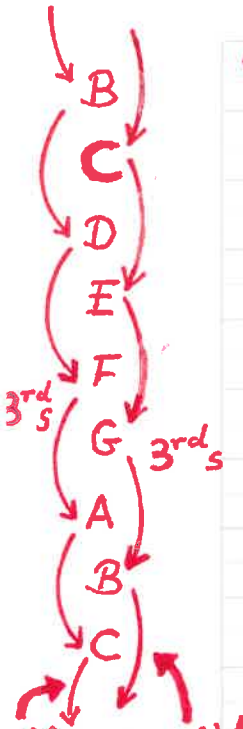
For completeness, the seven triads and their associated bit sequences are listed as well. Since the 3rd's and 5th's constituting the specific triads are also occurring in the corresponding 7th chords, the specific 3-bit sequences also show up as sub-sequences on the left-hand side.

THUS, TO RECOGNIZE A SPECIFIC TRIAD OR 7th CHORD FROM A SEQUENCE / GROUP / SET OF NOTES, ONE CAN USE THE 7-BIT SEQUENCE $b_1 b_2 b_3 b_4 b_5 b_6 b_7$ TO DETERMINE WHETHER THE SUBSET OF "ON-BITS" (= BITS WITH VALUE 1) IS POSSIBLY INDICATING THE EXISTENCE / USE OF A CERTAIN CHORD IN A COMPOSITION.

One must keep in mind that "ornamental," "trill," "transitional" or "chromatic-sequence" notes can arise as well.

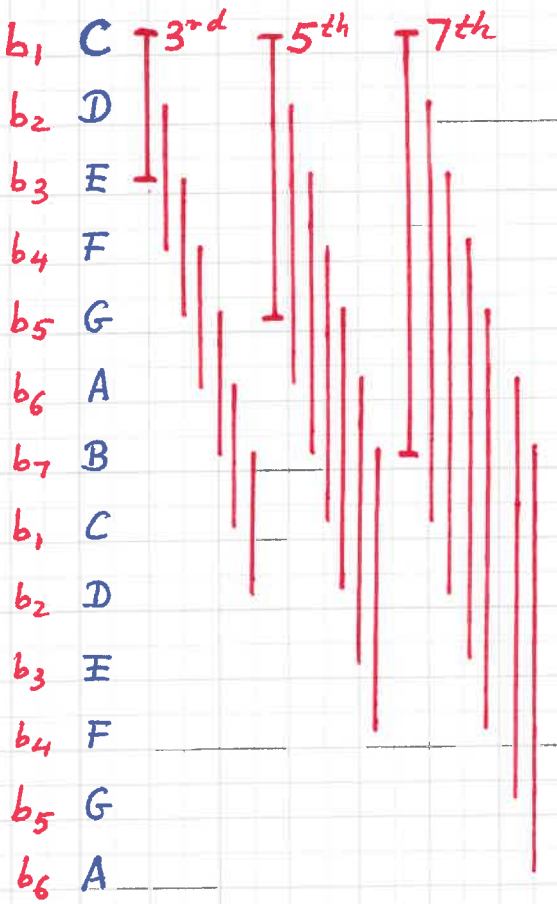
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• CHORD RECOGNITION FROM NOTES



Nearly all/most common chords can be understood or represented by "stacked 3rd's" – when ordering the chord's notes accordingly!

Note. Modifications of the "base notes" via '#' and 'b' are ignored in the discussed method for harmony / chord recognition and analysis. Thus, every "base note" stands as a representative note for its three forms, i.e., $C \hat{=} \{C^b, C, C^\#\}, \ldots, B \hat{=} \{B^b, B, B^\#\}$. Of course, to analyze all major and minor keys and their triads, 7th chords — including all key-inherent ones — properly, one must consider '#' and 'b' modifications.



Since the described method recognizes, for example, a triad by note sets, it does not matter whether the triad is played in root position or in an inversion: CEG, EGC, GCE all have the associated 7-bit "code" $b_1b_2b_3b_4b_5b_6b_7 = 1010100$. The left illustration focuses on the fact that triads and 7th chords "contain" 3rd's, 5th's and 7th's; all possibilities are shown.

Example:

I: CEG

CE CG EG

C E G

∅

⇒ also use II: DFA, ..., VII

Note. (Modifications via '#' and 'b' are not considered.) One must keep in mind the NUMBERS of POSSIBLE CHORDS: THE 3 (4) NOTES OF A TRIAD NOTE SET (7th CHORD NOTE SET) DEFINE 1 3-NOTE, 3 2-NOTE, 3 "1-NOTE" AND 1 "0-NOTE" CHORDS (1 4-NOTE, 4 3-NOTE, 6 2-NOTE, 4 "1-NOTE" AND 1 "0-NOTE CHORDS). THE 3(4) NOTES IMPLY $2^3 = 8$ ($2^4 = 16$) CHORDS.

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• CHORD RECOGNITION FROM NOTES

11/13/2024

It is therefore possible to understand the note sets defining certain chords via "index sets" — where the index value **1** is associated with the "base note". The following examples explain this viewpoint via the seven key-inherent triads and 7th chords for $C \hat{=} 1$:

I	$C_1\ E_3\ G_5$	1 3 5	These seven index
II	$D_2\ F_4\ A_6$	2 4 6	sets represent all
III	$E_3\ G_5\ B_7$	3 5 7	the listed seven
IV	$F\ A\ C_1$	1 4 6	triads (complete
V	$G\ B\ D$	2 5 7	triads) and also
VI	$A\ C\ E$	1 3 6	their two inversions.
VII	$B\ D\ F$	2 4 7	

(+3 between each pair of index rows)

I	$C_1\ E_3\ G_5\ B_7$	1 3 5 7	These seven index
II	$D_2\ F_4\ A_6\ C_1$	1 2 4 6	sets represent all
III	$E\ G\ B\ D$	2 3 5 7	the listed seven
IV	$F\ A\ C\ E$	1 3 4 6	7th chords (complete
V	$G\ B\ D\ F$	2 4 5 7	7th chords) and also
VI	$A\ C\ E\ G$	1 3 5 6	their three in-
VII	$B\ D\ F\ A$	2 4 6 7	versions.

(+1 between each pair of index rows)

- One must keep in mind that such chords can also appear in "incomple representation" — where some not are not used.
- The fact that such chords are constructed by "stacking 3rds" implies that two consecutive indices (of complete chords) generally have an "index distance" of **2**.
- Analogously, one must consider the notes D, E, F, G, A and B as "base notes" — having the associated index value **1** ($\hat{=}$ **I**).

• Visualizing Most Common Chords

11/20/2024

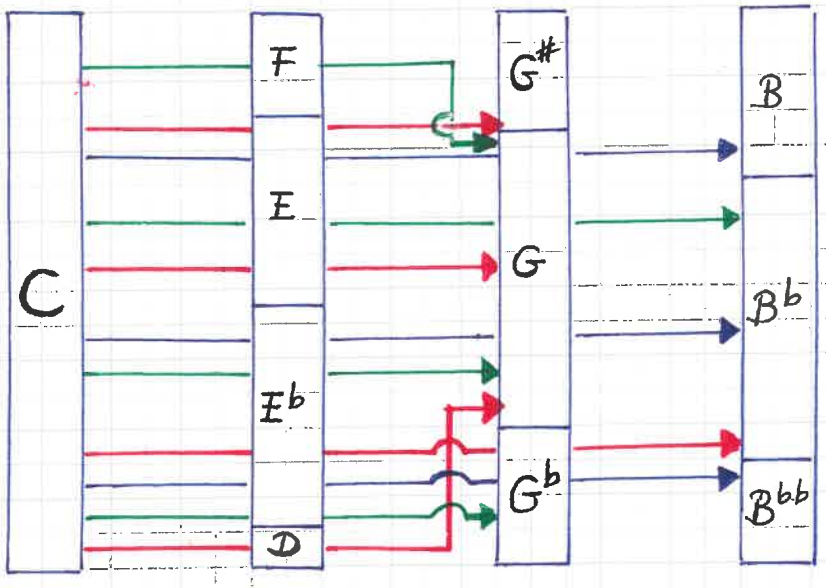
Common triads and 7th chords (over C)* are:

Name	Notes of chord, with notes of inversions		Notes of "general chords" represented via ordinal numbers relative to root note ≙ 1 (e.g., C≙1, D≙2, E≙3, F≙4, G≙5, B≙7)	
C sus4	C F G	C F	1 4 5	1 4
C aug	C E $G^{\#}$	C E	1 3 5^{+}	1 3
C maj 7	C E G B	C E G	1 3 5 7	1 3 5
C 7	C E G B^{b}	C E G	1 3 5 $^{-}7$	1 3 5
C maj	C E G	C E	1 3 5	1 3
C min 7	C E^{b} G B^{b}	C E^{b} G	1 $^{-}3$ 5 $^{-}7$	1 $^{-}3$ 5
C min	C E^{b} G	C E^{b}	1 $^{-}3$ 5	1 $^{-}3$
C halfdim 7	C E^{b} G^{b} B^{b}	C E^{b} G^{b}	1 $^{-}3$ $^{-}5$ $^{-}7$	1 $^{-}3$ $^{-}5$
C fulldim 7	C E^{b} G^{b} B^{bb}	C E^{b} G^{b}	1 $^{-}3$ $^{-}5$ $^{--}7$	1 $^{-}3$ $^{-}5$
C dim	C E G^{b}	C E	1 3 $^{-}5$	1 3
C sus2	C D G	C D	1 2 5	1 2

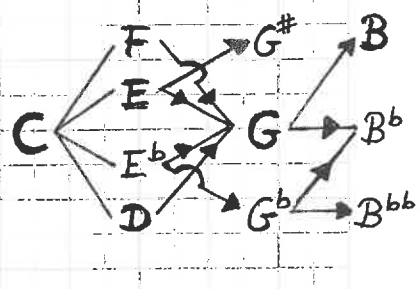
* Keep also in mind:
- Neapolitan
- Sixte Ajoutée chords

(-) ≙ 'b'
(+) ≙ '#'
(--) ≙ "bb"

⇒ Visualization:



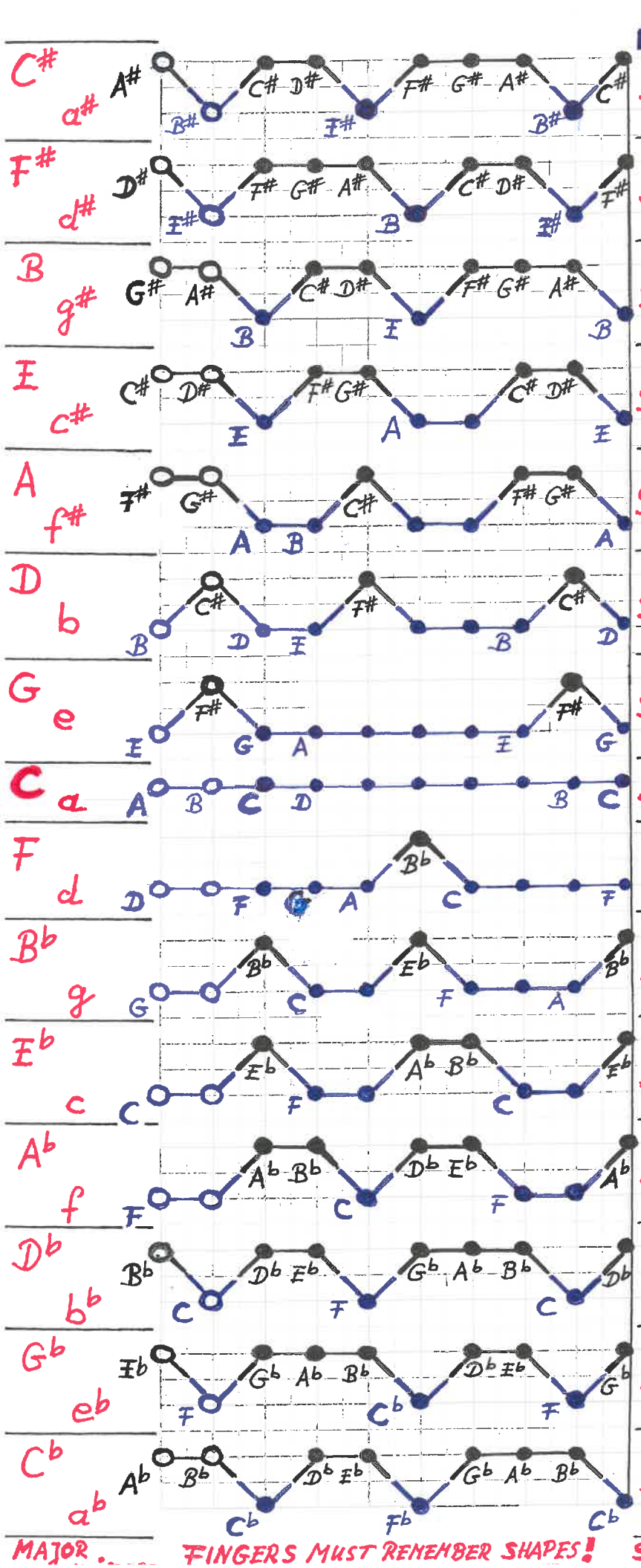
"Compact" simplification:



Last note of chord indicated by ▶, ▶, ▶.

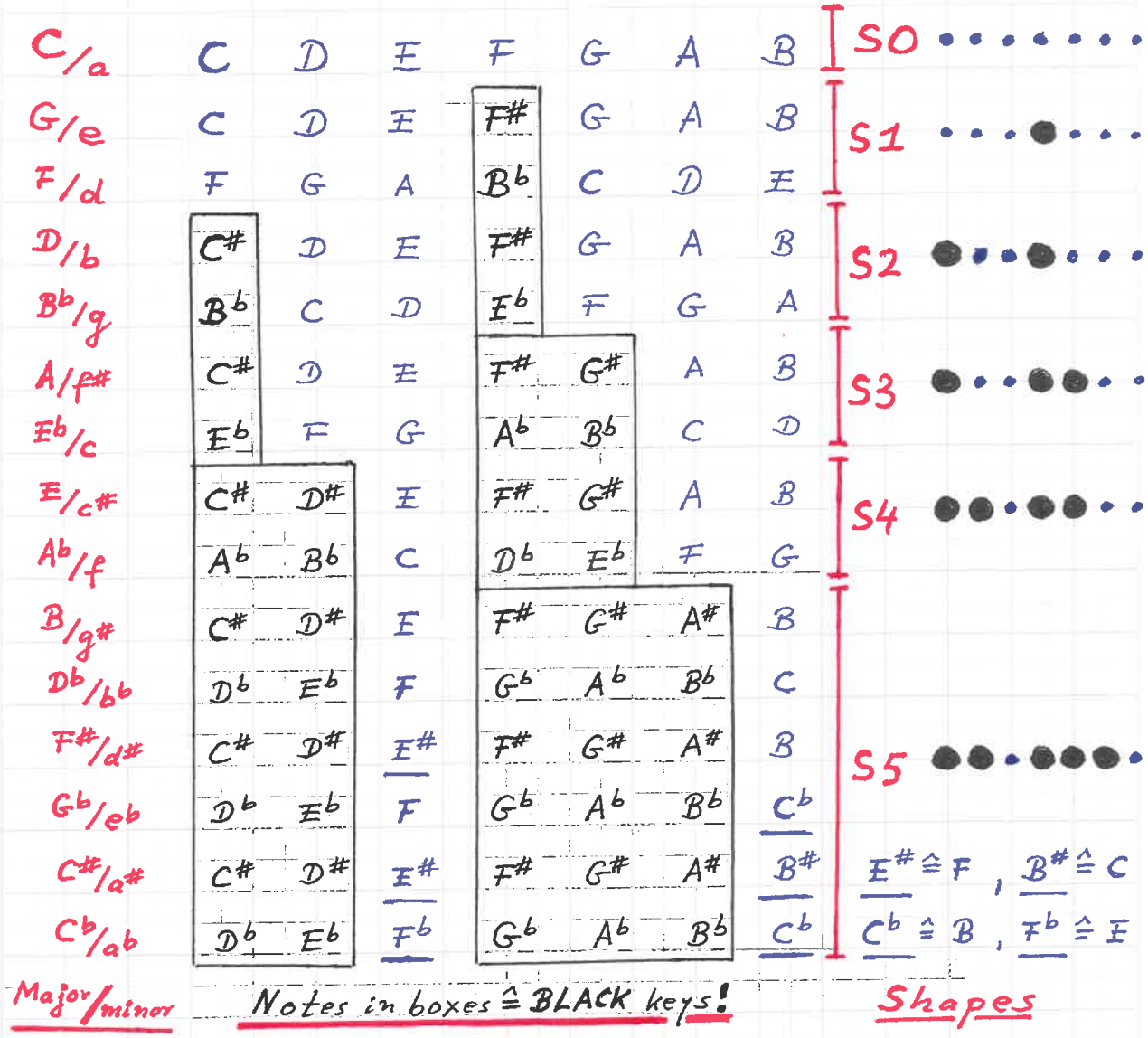
This representation applies to all 12 half-tones serving as root 1:

"$1 \in \{C, C^{\#}(D^{b}), D, D^{\#}(E^{b}), E, F, F^{\#}(G^{b}), G, G^{\#}(A^{b}), A, A^{\#}(B^{b}), B\}$."


Scale Shapes -I- 11/21/2024
C# a#
A# C# D# F# G# A# C# B# F# B#
F# d#
D# F# G# A# C# D# F# E# B E#
B g#
G# A# C# D# F# G# A# B E B
E c#
C# D# F# G# C# D# E A E
A f#
F# G# C# F# G# A B A
D b
C# F# C# B D E B D
G e
F# G A F# E G
C a
A B C D B C
F d
D F G A Bb C F
Bb g
G Bb C Eb F A Bb
Eb c
C Eb F Ab Bb C Eb
Ab f
F Ab Bb C Db Eb F Ab
Db bb
Bb C Db Eb F Gb Ab Bb C Db
Gb eb
Eb F Gb Ab Bb Cb Db Eb F Gb
Cb ab
Ab Bb Cb Db Eb Fb Gb Ab Bb Cb
S7#
S6#
S5#
S4#
S3#
S2#
S1#
S0
S1b
S2b
S3b
S4b
S5b
S6b
S7b
=
=
=
EQUAL
MAJOR
FINGERS MUST REMEMBER SHAPES!
Shape
Legend
WHITE keys are shown as 'o', BLACK keys are shown as '•'.
(The additional keys required for minor key signatures are shown as 'o' and 'o'.)
Scale shapes are called based on their numbers of '#' or 'b', i.e., S0, S1#, ..., S7#, S1b, ..., S7b. Since C#maj ≙ Dbmaj, F#maj ≙ Gbmaj and Bmaj ≙ Cbmaj, their shapes are equal: S7#=S5b, S6#=S6b, S5#=S7b.
Key-inherent chords, e.g., triads and 7th chords, are also defined by the shape of a specific key – by considering / playing the respective chord "subshapes".
The shapes S1b, ..., S7b are "shifted versions" of the shapes S1#, ..., S7#, respectively – "polygon shifts."
Shapes change gradually, "smoothly," from S0 to S7# and from S0 to S7b.

11/22/2024

Scale Shapes



■ By SHIFTING the notes of all the major and minor keys — such that the notes with associated black keys appear in the same columns, as much as possible — one sees the SIX SHAPES that define the six possible black-and-white-key sequences: S0 (no black keys), S1 (one black key), ..., S5 (five black key). For a given key, its associated scale shape repeats in all octaves of the 88-key keyboard.

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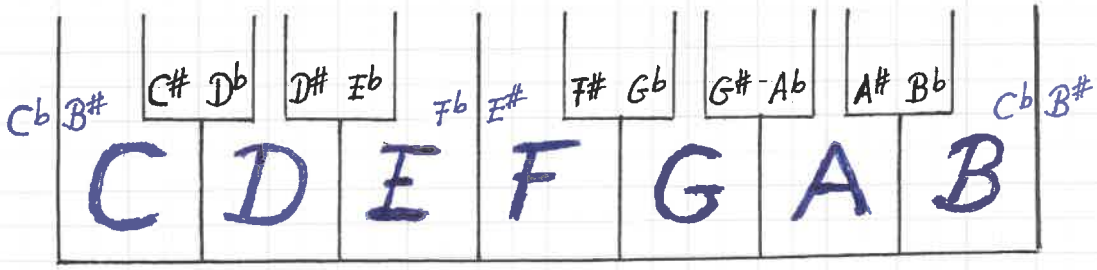
11/23/2024

Scale Shapes

An alternative way to organize the notes to represent scale shapes is the following table:

Major/minor								
C/a	F	G	A	B	C	D	E	S0
G/e	F#	G	A	B	C	D	E	S1
F/d	Bb	C	D	E	F	G	A	
D/b	F#	G	A	B	C#	D	E	S2
Bb/g	Eb	F	G	A	Bb	C	D	
A/f#	F#	G#	A	B	C#	D	E	S3
Eb/c	Ab	Bb	C	D	Eb	F	G	
E/c#	F#	G#	A	B	C#	D#	E	S4
Ab/f	Db	Eb	F	G	Ab	Bb	C	
B/g#	F#	G#	A#	B	C#	D#	E	
Db/bb	Gb	Ab	Bb	C	Db	Eb	F	
F#/d#	F#	G#	A#	B	C#	D#	E#	S5
Gb/eb	Gb	Ab	Bb	Cb	Db	Eb	F	
C#/a#	F#	G#	A#	B#	C#	D#	E#	
Cb/ab	Gb	Ab	Bb	Cb	Db	Eb	Fb	

Notes in boxes ≙ BLACK keys! Shapes

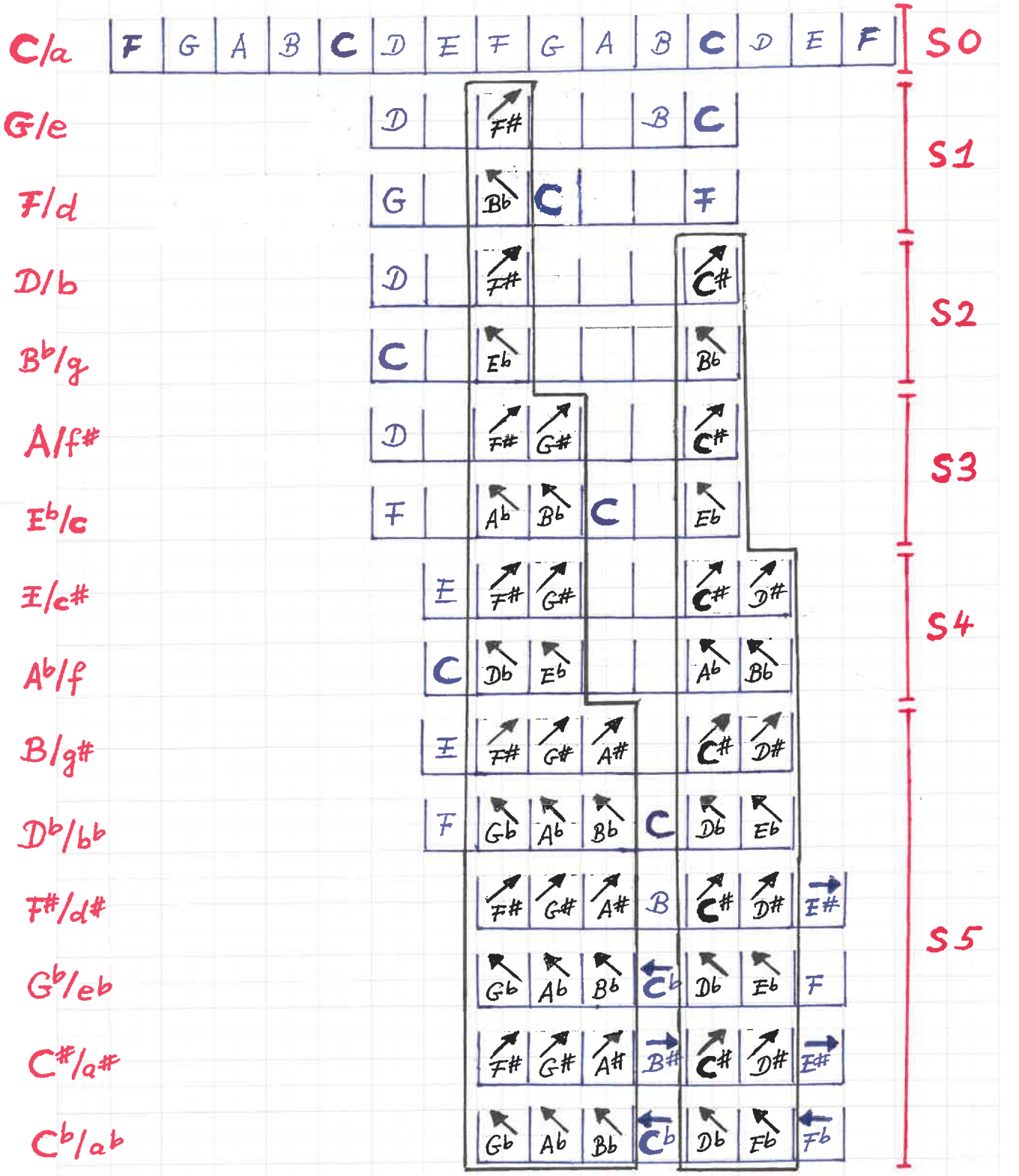


"One can always think of only the 'basic notes' C, D, E, F, G, A, B — modified by the key-specific '#' and 'b'.

11/24/2024

Scale Shapes

One can also visualize the notion of understanding all scales / scale shapes consisting only of the "seven basic notes (and keys)" as follows:

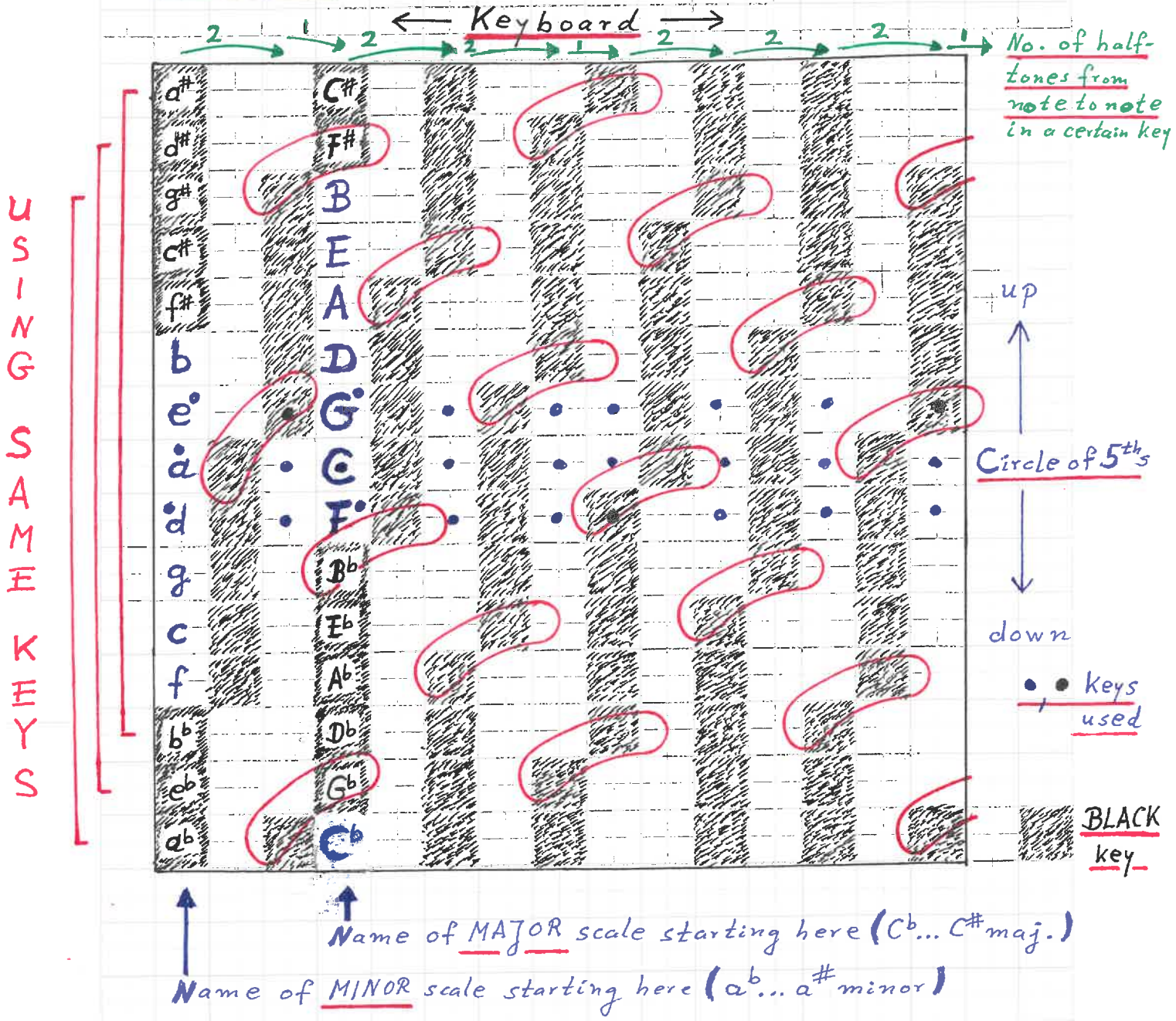


Here, the "seven-note keyboard" is shifted, leaving notes modified by '#' and 'b' in BLACK boxes.

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11/26/2024

Scale Shapes



- This visualization shows that "key movement one-by-one, following the circle of 5ths order," is the "smoothest possible movement from key to key: Only the key-pairs with a ( ) -high-lighted pair of notes/keys permit minimal change. For example, key modulation following the circle of 5ths order requires one to only change the use of black keys minimally."

• KEYS, SCALES, CIRCLES

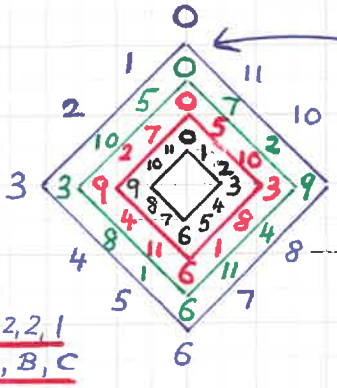
12/2/2024

One can analyze key signatures, implied scales and "circles," e.g., the circle of 5ths, using a viewpoint from combinatorics. The following analysis considers half-tone steps as basis for indexing and numbering. Further, the index of a root note is 0, and (most) calculations use "modulo-12 arithmetic." We do not use '#' or 'b' notation; instead, for example, we write ^{F}G to represent $F^{\#}$ and G^{b}. The next sequences of notes are obtained when starting with the root note C (=0) and ending with the note C after "moving up" from note to note by a constant number of half-tone steps, called s:

s=1	0	1	2	3	4	5	6	7	8	9	10	11	0
	C	^{C}D	D	^{D}E	E	F	^{F}G	G	^{G}A	A	^{A}B	B	C
s=2	0	2	4	6	8	10	0						
	C	D	E	^{F}G	^{G}A	^{A}B	C						
s=3	0	3	6	9	0								
	C	^{D}E	^{F}G	A	C								
s=4	0	4	8	0									
	C	E	^{G}A	C									
s=5 ('b')	0	5	10	3	8	1	6	11	4	9	2	7	0
	C	F	^{A}B	^{D}E	^{G}A	^{C}D	^{F}G	B	E	A	D	G	C
s=6	0	6	0										
	C	^{F}G	C										
s=7 ('#')	0	7	2	9	4	11	6	1	8	3	10	5	0
	C	G	D	A	E	B	^{F}G	^{C}D	^{G}A	^{D}E	^{A}B	F	C
s=8	0	8	4	0									
	C	^{G}A	E	C									

(number of half-tone steps = $\frac{1}{2}\cdot s$)

Sequence of numbers of half-tone steps for C maj.: 2,2,1,2,2,2,1
⇒ C, D, E, F, G, A, B, C



Four circles for number of half-tone steps s=1, s=5, s=7, s=11, MODULO 12.

MOVING FORWARD / MOVING BACKWARD → IN CIRCLE OF FIFTHS

s=5 ('b'): C, F, B^{b}, E^{b}, A^{b}, D^{b}, G^{b}, C^{b}

s=7 ('#'): C, G, D, A, E, B, $F^{\#}$, $C^{\#}$

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• KEYS, SCALES, CIRCLES

s=9	0	9	6	3	0								
	C	A	F_G	D_E	C								
s=10	0	10	8	6	4	2	0						
	C	A_B	G_A	F_G	E	D	C						
s=11	0	11	10	9	8	7	6	5	4	3	2	1	0
	C	B	A_B	A	G_A	G	F_G	F	E	D_E	D	C_D	C
s=0 (s=12)	0	0											
	C	C											

• **Note.** Instead of using positive integer values for s, with $0 \leq s \leq 11$, one can also use NEGATIVE integer values for s, with $-11 \leq s \leq 0$. These are "corresponding positive and negative values" for s: $s=1 \hat{=} s=-11$; $s=2 \hat{=} s=-10$; $s=3 \hat{=} s=-9$; $s=4 \hat{=} s=-8$; $s=5 \hat{=} s=-7$; $s=6 \hat{=} s=-6$; $s=7 \hat{=} s=-5$; $s=8 \hat{=} s=-4$; $s=9 \hat{=} s=-3$; $s=10 \hat{=} s=-2$; $s=11 \hat{=} s=-1$. Thus, a specific positive s-value has the corresponding negative s-value of $s-12$: $s \hat{=} s-12$, where $s \in \{0, 1, \ldots, 11\}$.

(Example: Using $s=-5$, one obtains the index/number sequence $0, -5, -10, -15, -20, -25, \ldots, -60$ that is equivalent to $0, -5 \bmod 12, -10 \bmod 12, -15 \bmod 12, \ldots, -60 \bmod 12$, which becomes **0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, 0** — which is the sequence one also obtains for $s = -5+12 = 7$. In other words, "moving from C by -5 half-tone steps again and again corresponds to / is equivalent to moving from C by +7 half-tone steps again and again.") As a consequence, one sees that the sequences for s=1 and s=-11 are 0, 1, 2, ..., 0; for s=11 and s=-1 are 0, 11, 10, ..., 0; for s=5 and s=-7 are 0, 5, 10, ..., 0; and for s=7 and s=-5 are 0, 7, 2, ..., 0.

• KEYS, SCALES, CIRCLES

12/3/2024

	S=1: C	C_D	D	D_E	E	F	F_G	G	G_A	A	A_B	B	C
s=0	0 <												12(0)
s=1	0 <	1 <	2 <	3 <	4 <	5 <	6 <	7 <	8 <	9 <	10 <	11 <	12(0)
s=2	0 <		2 <		4 <		6 <		8 <		10 <		12(0)
s=3	0 <			3 <			6 <			9 <			12(0)
s=4	0 <				4 <				8 <				12(0)
s=5	12(0) >	5 <	10 >	3 <	8 >	1 <	6 <	11 >	4 <	9 >	2 <	7 >	0(12)
s=6	0(12) ≶						6 ≶						12(0)
s=7	0(12) <	7 >	2 <	9 >	4 <	11 >	6 >	1 <	8 >	3 <	10 >	5 <	12(0)
s=8	12(0) >				8 >				4 >				0(12)
s=9	12(0) >			9 >			6 >			3 >			0(12)
s=10	12(0) >		10 >		8 >		6 >		4 >		2 >		0(12)
s=11	12(0) >	11 >	10 >	9 >	8 >	7 >	6 >	5 >	4 >	3 >	2 >	1 >	0(12)
s=12 (0)	12(0) >												0(12)

specific chromatic note sequence and index sequence

EQUIVALENT

6 '>', 6 '<'

'6' = tritonus

6 '<', 6 '>'

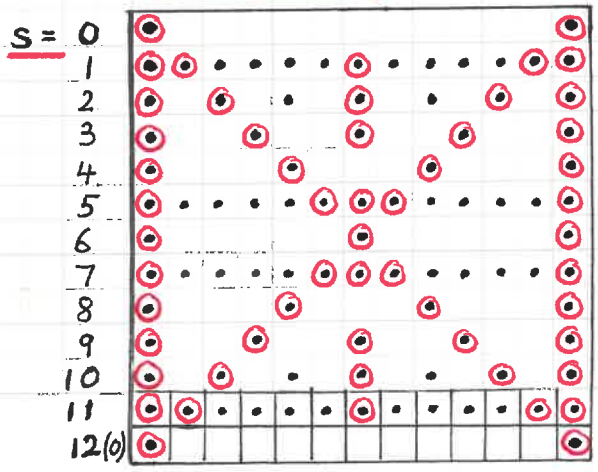
s = 0, 1, 2, 3, 4 : only '<' ⇒ moving up in octave

s = 8, 9, 10, 11, 12 : only '>' ⇒ moving down in octave

s = 5, 7 : '>' and '<' ⇒ moving up/down in octave

s = 6 : only '<' or only '>' ⇒ moving up or down in octave

(All "movements" from note to note are restricted to only one octave as a consequence of modulo-12 arithmetic, allowing only 0, 1, 2, ..., 12 half-tone steps from C.)

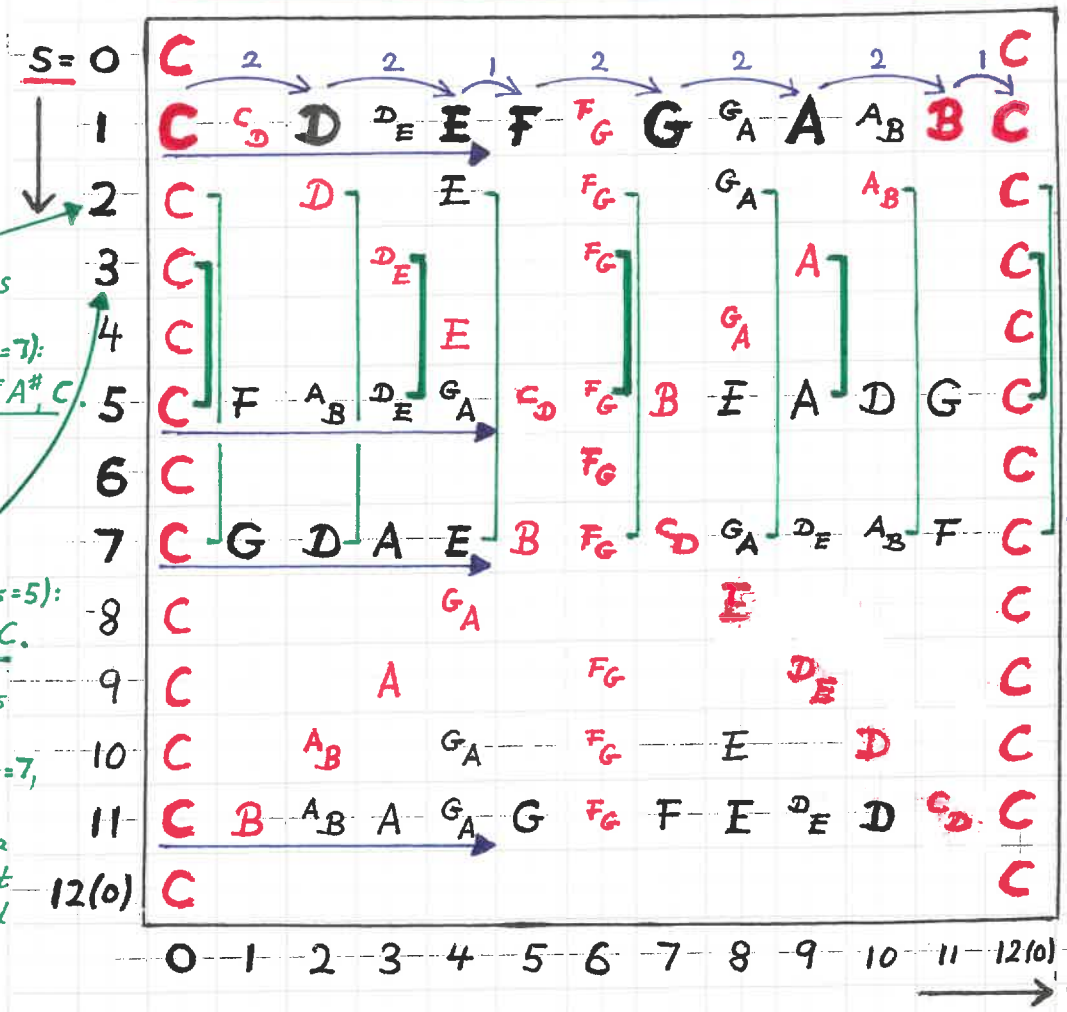


The left figure shows the abstract "number-of-half-tone-steps" pattern defined by the above list of the 13 (s=0...12) index and implied note sequences. The figure of the "bi-periodic pattern" shows vertical, diagonal and horizontal (c-c) symmetry axes.

• KEYS, SCALES, CIRCLES

12/3/2024

TABLE OF THE 13 CIRCLES WITH ROOT C



Chromatic scale "up": C, C♯, D, D♯, E, F, ...

Circle of 2nd's

Octave = 4 minor 3rd's

Octave = 3 major 3rd's

Circle of 5th's "backward": C, F, B♭, E♭, A♭, D♭, ...

Octave = 2 tritones

Circle of 5th's "forward": C, G, D, A, E, B, ...

Octave = 3 major 3rd's

Octave = 4 minor 3rd's

Circle of 2nd's (reverse)

Chromatic scale "down": C, B, B♭, A, A♭, G, ...

- s=2 leads to a note subset (of s=7): C, D, E, F♯, G♯, A♯, C.
- s=3 leads to a note subset (of s=5): C, E♭, G♭, A, C.
- s=4 leads to a note subset of s=7, and s=6 leads to a note subset of s=5 and s=7.

⇒ THIS TABLE OF NOTES — THE NOTES OF THE 13 CIRCLES WITH ROOT C — REPRESENTS ALL NOTES IN A TWO-DIMENSIONAL TABLE $T_{s,n}$, $s = 0 \dots 12$, $n = 0 \dots 12$. THE TABLE ILLUSTRATES AN AMAZING SYMMETRY DEFINED BY AND CHARACTERIZING THE CIRCLES:

$$T_{s,n} = T_{12-s,\,12-n} \quad , \quad s, n = 0 \dots 12 \, .$$

(Examples: $T_{1,1} = C_D = T_{11,11}$; $T_{5,3} = D_E = T_{7,9}$ ..)

Thus, this "system of notes," this table of circles, has POINT SYMMETRY with respect to the "note center" F_G — the tritonus for root C.

12/4/2024

• KEYS, SCALES, CIRCLES

The table of the 13 circles with root a — using small letters to indicate MINOR keys — can be constructed analogously:

S=														
0	a	2		1		2		2		1		2	2	a
1	a	a_b	b	c	c_d	d	d_e	e	f	f_g	g	g_a	a	Chromatic-up
2	a		b		c_d		d_e		f		g		a	
3	a			c			d_e			f_g			a	
4	a				c_d				f				a	
5	a	d	g	c	f	a_b	d_e	g_a	c_d	f_g	b	e	a	Circle of 5ths – backward
6	a						d_e						a	
7	a	e	b	f_g	c_d	g_a	d_e	a_b	f	c	g	d	a	Circle of 5ths – forward
8	a				f				c_d				a	
9	a			f_g			d_e			c			a	
10	a		g		f		d_e		c_d		b		a	
11	a	g_a	g	f_g	f	e	d_e	d	c_d	c	b	a_b	a	Chromatic-down
12(0)	a												a	

$T_{s,n}$ 0 1 2 3 4 5 6 7 8 9 10 11 12(0) → note (n)

⇒ This table also exhibits symmetry:

$$T_{s,n} = T_{12-s,\,12-n}\,, \quad s,n = 0 \ldots 12\,.$$

⇒ This table can be constructed for any root note, for all major and minor keys: ..., B^b, F, C, G, D, ...; ..., g, d, a, e, b, ...

• The circle of 5ths and the note-associated indices/numbers (in modulo-12 arithmetic) are especially important:

C	G	D	A	E	B	F_G	C_D	G_A	D_E	A_B	F	C
0	7	2	9	4	11	6	1	8	3	10	5	0

"Moving from C (root) with index 0 by 7 half-tone steps – again and again"

• KEYS, SCALES, CIRCLES 12/5/2024

It is possible to "analyze and understand" the relationship between the circle of 5ths and scales by considering the note-assigned indices from the table on the previous page (bottom table) and note-assigned indices following the scale-implied order of notes – using modulo-12 arithmetic.

In the following, we use the following indexing / numbering of the notes of the chromatic sale on C:

C=0(12), $^{C}_{D}$=1, D=2, $^{D}_{E}$=3, E=4, F=5, $^{F}_{G}$=6, G=7, $^{G}_{A}$=8, A=9, $^{A}_{B}$=10, B=11, C=12(0). We can thus use these numbers to list the notes of the scales of all major keys, for example:

C	0	2	4	5	7	9	11	0	C=B#
G	7	9	11	0	2	4	6	7	
D	2	4	6	7	9	11	1	2	
A	9	11	1	2	4	6	8	9	
E	4	6	8	9	11	1	3	4	E=Fb
B	11	1	3	4	6	8	10	11	B=Cb
$^{F}_{G}$	6	8	10	11	1	3	5	6	F#=Gb
$^{C}_{D}$	1	3	5	6	8	10	0	1	C#=Db
$^{G}_{A}$	8	10	0	1	3	5	7	8	G#=Ab
$^{D}_{E}$	3	5	7	8	10	0	2	3	D#=Eb
$^{A}_{B}$	10	0	2	3	5	7	9	10	A#=Bb
F	5	7	9	10	0	2	4	5	F=E#
+		2	2	1	2	2	2	1	

The left table shows the scale indices for all 12 major scales. Analogously, one can construct the table for all 12 minor scales:

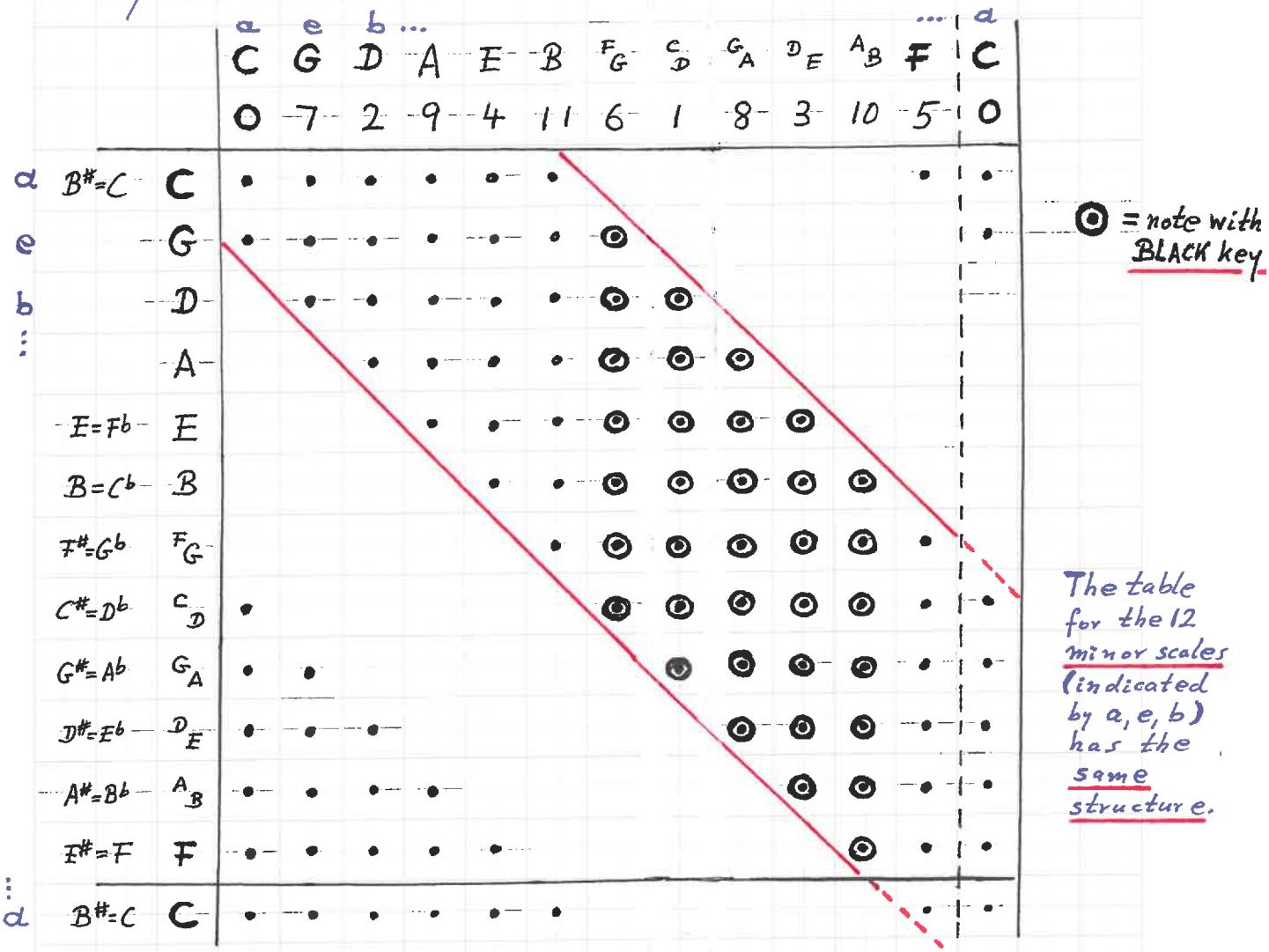
a	9	11	0	2	4	5	7	9
e	4	6	7	9	11	0	2	4
b	11	1	2	4	6	7	9	11
f#	6	8	9	11	1	2	4	6
c#	1	3	4	6	8	9	11	1
g#	8	10	11	1	3	4	6	8
d#	3	5	6	8	10	11	1	3
a#	10	0	1	3	5	6	8	10
f	5	7	8	10	0	1	3	5
c	0	2	3	5	7	8	10	0
g	7	9	10	0	2	3	5	7
d	2	4	5	7	9	10	0	2
+		2	1	2	2	1	2	2

Based on these indices and index tables, one can construct a table that nicely captures the relationship between the circle of 5ths and the notes of all 12 keys, see next page.

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• Keys, Scales, Circles 12/5/2024

In the following table we place a mark in a specific place when the associated note belongs to the scale of the key being considered. The order of the rows follows the circle-of-5ths-implied order of keys; the order/indices of the columns are defined by the note indices of the circle-of-5ths notes:



This table perfectly shows the periodic/cyclic nature of the notes belonging to the scales of the 12 keys — when using the circle of 5ths as basis for "ordering." Further, the visible diagonal "band structure" of the table shows that moving from C to G, from G to D, from D to A etc. is "optimal" — since only one note changes each time.

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• Keys, Scales, Circles 12/6/2024

A main goal for analyzing the relationships between keys and scales is a "simple understanding" of the numbers and keyboard locations of the BLACK keys to be played. A combinatorics-based viewpoint combined with a modulo-12 arithmetic could be helpful for this purpose. The table on p. 6 (12/5/2024) can be used to define the implied pattern of white and black keys to be used for playing (○=white, ●=black):

	NOTE 1st	2nd	3rd	4th	5th	6th	7th	WHITE(○): 0,2,4,5,7,9,11 BLACK(●): 1,3,6,8,10							NO.'●'
C	0	2	4	5	7	9	11	○	○	○	○	○	○	○	0
G	7	9	11	0	2	4	6	○	○	○	○	○	○	●	1
D	2	4	6	7	9	11	1	○	○	●	○	○	○	●	2
A	9	11	1	2	4	6	8	○	○	●	○	○	●	●	3
E	4	6	8	9	11	1	3	○	●	●	○	○	●	●	4
B	11	1	3	4	6	8	10	○	●	●	○	●	●	●	5
F_G	6	8	10	11	1	3	5	●	●	●	○	●	●	○	5
C_D	1	3	5	6	8	10	0	●	●	○	●	●	●	○	5
G_A	8	10	0	1	3	5	7	●	●	○	●	●	○	○	4
D_E	3	5	7	8	10	0	2	●	○	○	●	●	○	○	3
A_B	10	0	2	3	5	7	9	●	○	○	●	○	○	○	2
(E#) F	5	7	9	10	0	2	4	○	○	○	●	○	○	○	1
(B#) C	0	2	4	5	7	9	11	○	○	○	○	○	○	○	0

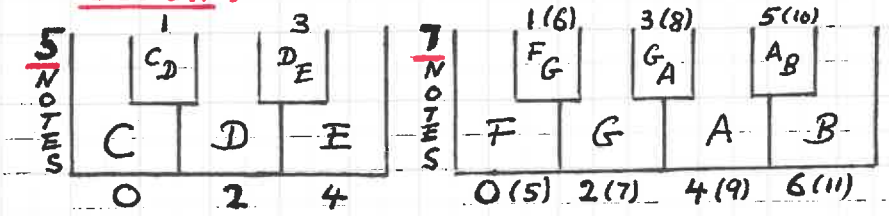
The right part of this table identifies the black keys to be played for the 12 major keys (and the 12 parallel minor keys). Considering the total of 24 major and (parallel) minor keys, 2 keys (C and a) do not use black keys; 4 keys use 1, 4 keys use 2, 4 keys use 3, 4 keys use 4, and 6 keys use 5 black keys.

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• Keys, Scales, Circles 12/6/2024

Notes. Many other concepts and approaches can be considered to study and analyze keys, scales, circles and the relationship to "meaningful and helpful rules" for memorizing the specific white and black keyboard keys to be used for playing. A few potential viewpoints are summarized next.

• The octave C-C consists of a 5-note and 7-note "block": C-$^{C}_{D}$-D-$^{D}_{E}$-E and F-$^{F}_{G}$-G-$^{G}_{A}$-A-$^{A}_{B}$-B.



The two blocks are shown in the left figure. Each block uses its own "local indexing," 0-4 and 0-6, respectively. Thus, viewing the blocks independently, the first is defined by modulo-5 and the second by modulo-7 arithmetic — when adopting a periodic viewpoint. When joining the blocks, the modulo-12 character results, as expected for the octave. Is this helpful for memorizing patterns?

• On page 4 (12/3/2024), we show that one obtains the circle of 5th for s=7 (forward) and s=5 (backward). This fact must be related to the two blocks described above.

• The keys with 0 (1; 2; 3; 4; 5) BLACK keyboard keys are C and a (G, F, e and d; D, $^{A}_{B}$, b and g; A, $^{D}_{E}$, $^{f}_{g}$ and c; E, $^{G}_{A}$, $^{c}_{d}$ and f; B, $^{F}_{G}$, $^{C}_{D}$, $^{g}_{a}$, $^{d}_{e}$, $^{a}_{b}$).

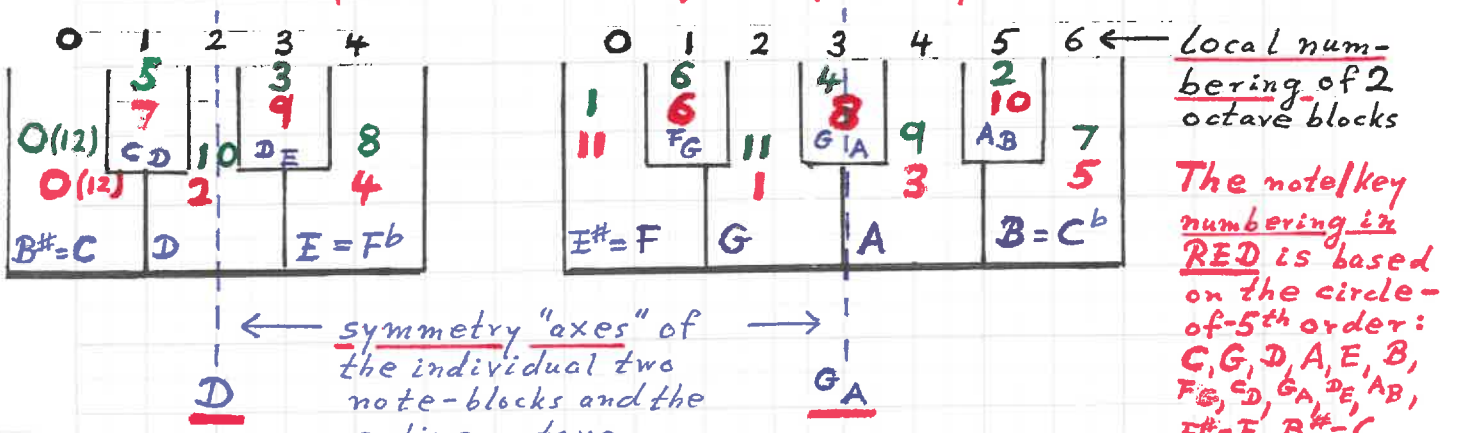
• The keyboard has 2 "symmetry axes": the key for D and the key for $^{G}_{A}$ (see above figure). Is this helpful for memorizing?

• TRANSLATING/TRANSPOSING, SCALING, REFLECTING are transformations potentially applicable to notes, scales, chords, keys, keyboard keys, note indices etc. — Ex.: SCALE (C, G, D, A, E, ...) = C, D, E, ... ≙ circle of 5ths ↦ circle of 9ths. Also: "Krebs" ⇒ forward ⇔ backward. Helpful?

Keys, Scales, Circles

12/7/2024

- One can use the partition of the octave C–C into a 5-note block $C-{}^{C}D-D-{}^{D}E-E$ and 7-note block $F-{}^{F}G-G-{}^{G}A-A-{}^{A}B-B$ to describe the sequential, one-by-one introduction of '#' and 'b' notes — and thus the sequences of introduced black keyboard keys — when using the circle of 5ths to define the order of introduction ($s=7$: forward, $s=5$ backward).



← local numbering of 2 octave blocks

The note/key numbering in RED is based on the circle-of-5th order: $C, G, D, A, E, B, {}^{F}G, {}^{C}D, {}^{G}A, {}^{D}E, {}^{A}B, E^{\#}=F, B^{\#}=C$

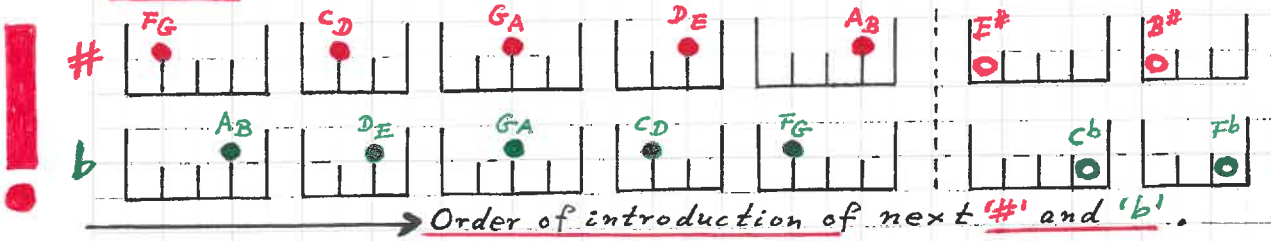
The numbering in GREEN is based on the order $C, F, {}^{A}B, {}^{D}E, {}^{G}A, {}^{C}D, {}^{F}G, C^{b}=B, F^{b}=E, B^{bb}=A, E^{bb}=D, A^{bb}=G$

⇒ Circle of 5ths forward/backward ≙ RED/GREEN nos.

From a GEOMETRICAL keyboard viewpoint, the keys with RED indices 7, 9, 12(0) are mapped to the keys with GREEN indices 3, 5, 8 via reflection by D.

From a GEOMETRICAL keyboard viewpoint, the keys with indices 6, 8, 10, 11 are mapped to the keys with GREEN indices 2, 4, 6, 7 via reflection of the RED-index keys 6, 8, 10, 11 by "axis" G_A.

⇒ ONE CAN UNDERSTAND AND MEMORIZE THE INTRODUCTION OF '#' AND 'b' — AND THUS OF BLACK KEYS — BY APPLYING '#' AND 'b' IN ALTERNATING FASHION TO THE 7- AND 5-NOTE BLOCKS:



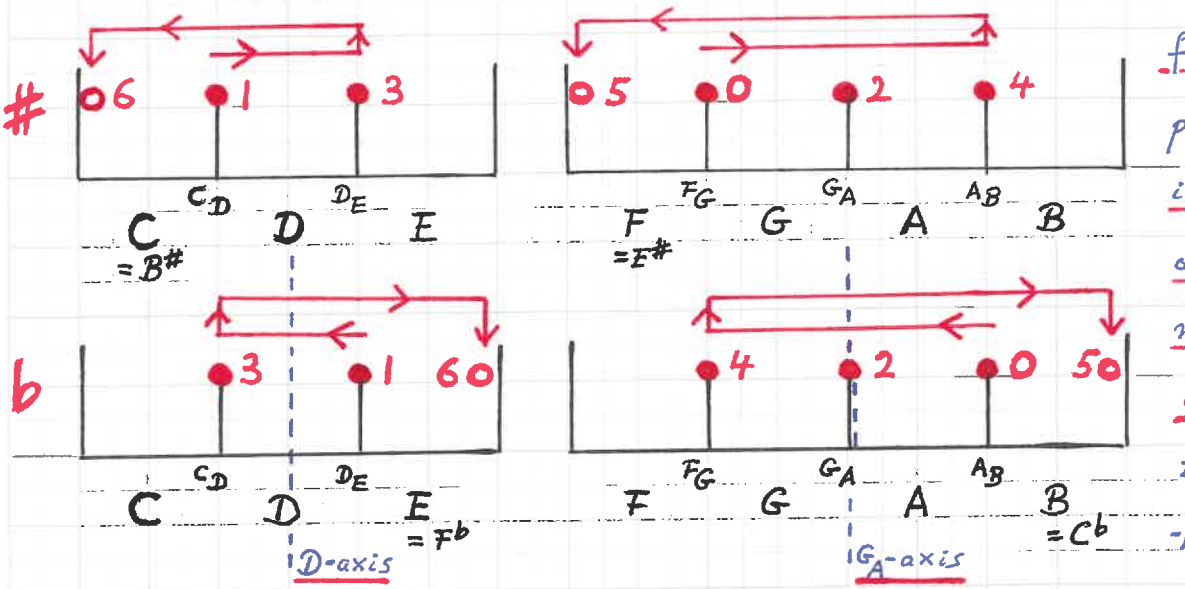
Note. The minor key scales/keys are a "by-product."

"Introducing '#'s ('b's) introduces BLACK KEYS from right to left (left to right), relative to the 7- and 5-note blocks.

• Keys, Scales, Circles 12/8/2024

"The main goal" when analyzing keys and the introduction of '#' and 'b' notes is the understanding of the implied, resulting patterns and sequences of keys on the keyboard – the "physical geometry." Ideally, one can identify a minimal number of patterns and sequences to be memorized.

Again, we consider the sequences of '#'- and 'b'-note introductions by the key sequences C-G-D-A-F-B-F#C# and C-F-B♭-E♭-A♭-D♭-G♭-C♭. These are the patterns:



The left figures represent the introductions of '#'- and 'b'-notes via RED cycles above the 5- and 7-key blocks.

The '#' cycles have counter-clockwise orientation, starting with a left-to-right keyboard movement.

The 'b' cycles have clockwise orientation, starting with a right-to-left keyboard movement.

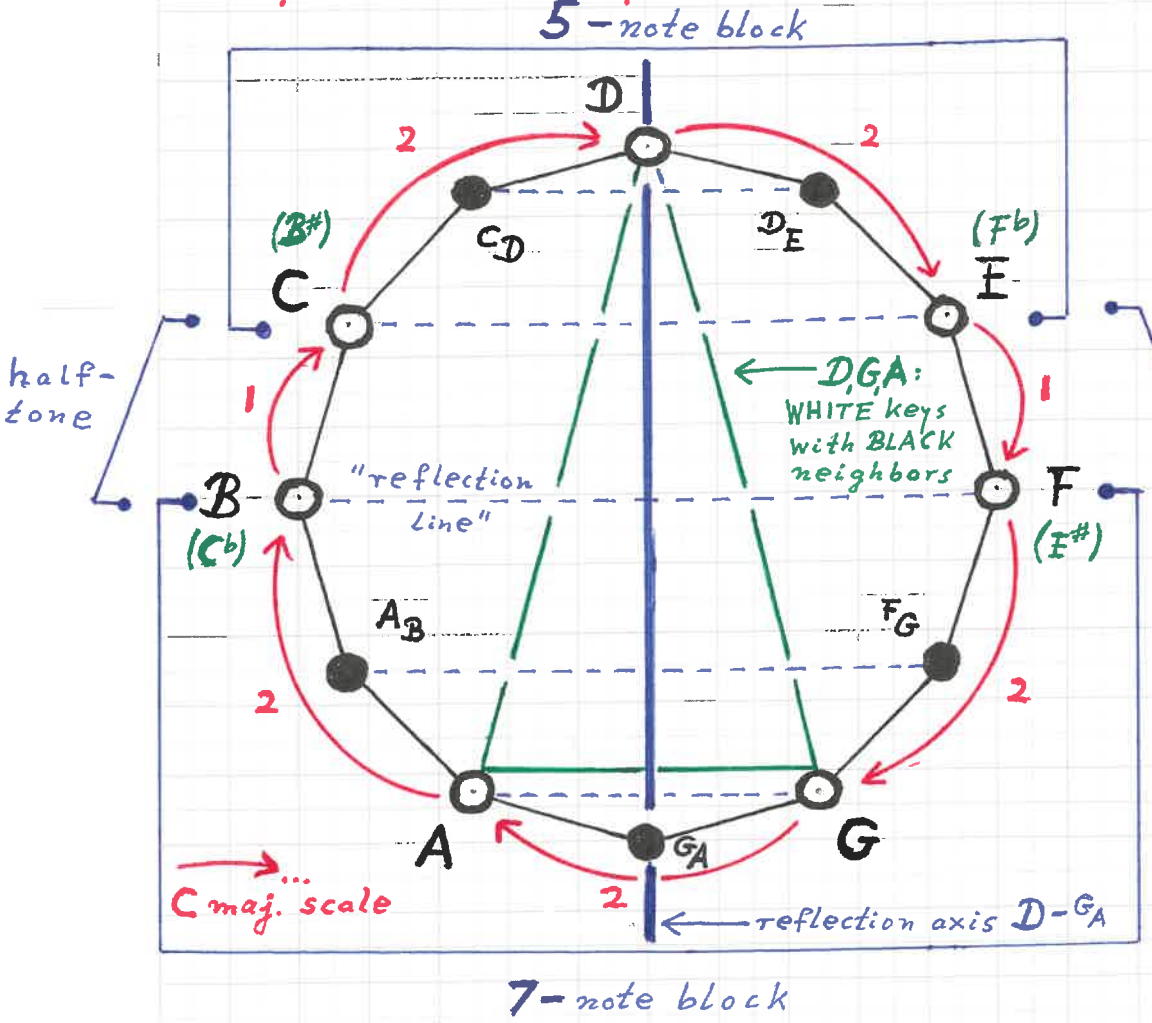
⇒ The two relevant sequences to memorize are $F_G - C_D - G_A - D_E - A_B - E^{\#}(F) - B^{\#}(C)$ for the introductions of '#'s and $A_B - D_E - G_A - C_D - F_G - C^{b}(B) - F^{b}(E)$ for the introductions of 'b's.

The seven indices used for both sequences are identical: 0, 1, 2, 3, 4, 5, 6. One can view the 'b' cycles as the result of "reflecting the '#' cycles against the D- and G_A-axis." Thus, one only needs to memorize the '#' or 'b' cycles.

• Keys, Scales, Circles

12/10/2024

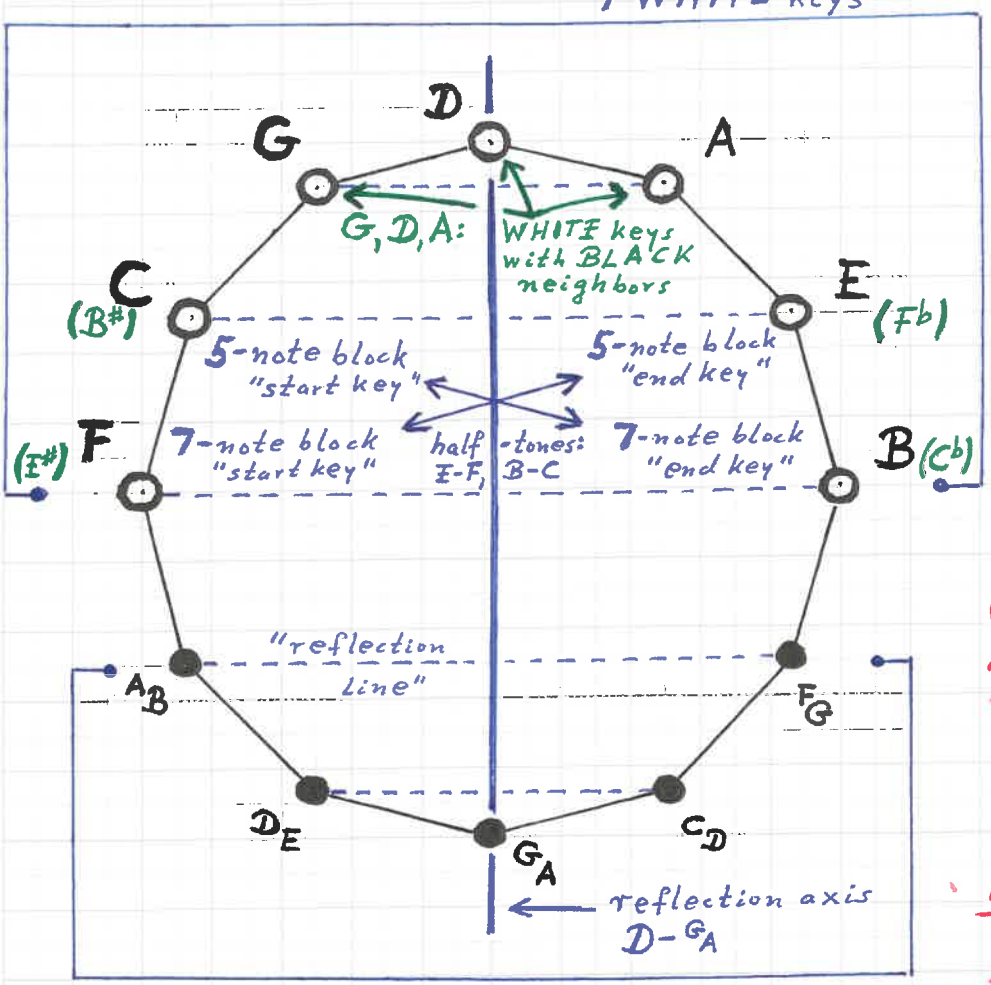
A circlar visualization of 12 halftones/notes defining the movement from C (0) to C (12) by "incrementing" by either 12 halftone steps (chromatic scale) or by 12 steps involving 7 halftone "jumps" at each step (circle of 5ths) makes it possible to clearly see SYMMETRIES. The goal must therefore be to relate these visible "geometrical symmetries" to the physical linear layout of the keyboard and to use them for memorization.



The left figure represents the linear keyboard octave C-C in a circular layout. It uses the 12 halftone steps to move from C to C. WHITE keys are visualized by 'O' and BLACK keys by '●'. The figure shows that the WHITE-BLACK key sequences exhibit symmetry relative to the "reflection axis" D-G_A, with D being the "center key" of the 5-note block and G_A being the "center key" of the 7-note block. HOW CAN ONE USE THIS VISUALIZATION TO MEMORIZE PATTERNS?

• Keys, Scales, Circles

12/10/2024



The left figure shows the circle of 5th's. The "scales with root notes having an associated WHITE key" are next to each other (top 7 notes); the "scales with root notes having an associated BLACK key" are also next to each other (bottom 5 notes [≙ 5 '#', 5 'b']).

This visualization also shows SYMMETRY, relative to the reflection axis D-G_A. As before, a note with an associated WHITE key (BLACK key) is reflected to a note that also has an associated WHITE key (BLACK key).

HOW CAN ONE USE THIS VISUALIZATION WITH ITS "SYMMETRY AND STRUCTURE" TO UNDERSTAND AND MORE EASILY MEMORIZE "GEOMETRY, SHAPES AND PATTERNS" OF SCALES, CHORDS ETC. ON THE LINEAR KEYBOARD?

One can analyze keys, scales and cyclic "behavior and geometry" using many other approaches — e.g., determining the "complementary nature" of pairs of scales (one having '#'s, one having 'b's) or the "opposite directions" of '#' and 'b' introductions (left-to-right, right-to-left).

• Keys, Scales, Circles 12/11/2024

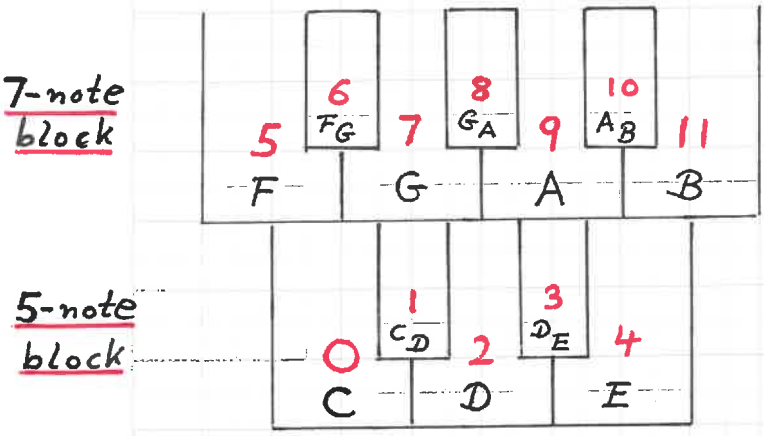
In the following, keys, scales, key-specific '#'s and 'b', and the order / direction of '#' and 'b' introductions are reviewed — when using the circle-of-5th progression order. Further, one can view specific keys and scales — and their corresponding keys on the keyboard — as being "complementary" to each other. Again, the goal of such analyses and characterizations of keys and scaled is the development of "simple rules" one can easily memorize for playing.

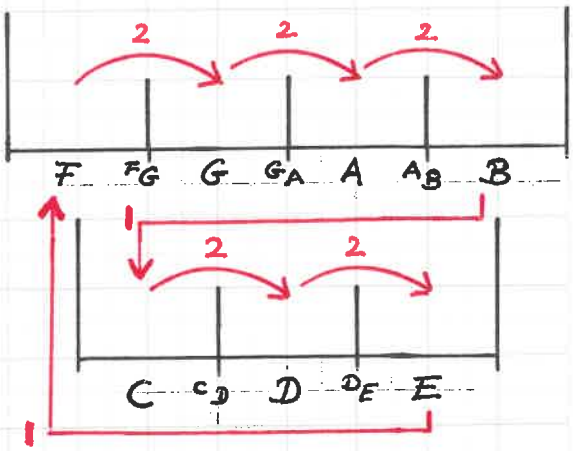
! ON THE PHYSICAL KEYBOARD, THE 5-NOTE AND 7-NOTE BLOCKS — C, ^{C}D, D, ^{D}E, E AND F, ^{F}G, G, ^{G}A, A, ^{A}B, B — ARE NEXT TO EACH OTHER, ARRANGED IN A "LINEAR FASHION." ALTERNATIVELY — AT LEAST CONCEPTUALLY — ONE CAN ARRANGE THE TWO BLOCKS "ON TOP EACH OTHER": THE 5-NOTE BLOCK CAN BE PLACED DIRECTLY UNDER THE 7-NOTE BLOCK — OR VICE VERSA — ANALOGOUSLY TO PLACING SEVERAL ORGAN MANUALS ON TOP OF EACH OTHER. THIS CONCEPT IS EXTREMELY HELPFUL FOR AN "EFFECTIVE VISUALIZATION AND UNDERSTANDING" OF THE RELATIONSHIPS BETWEEN KEYS, SCALES, CIRCLES AND THE PHYSICAL KEYS ON THE KEYBOARD. SPECIFICALLY, THIS CONCEPT GREATLY SIMPLIFIES THE STUDY OF THESE CHARACTERISTICS: ORDER OF INTRODUCTION OF '#'S AND 'b'S AND IMPLIED DIRECTION / ORIENTATION OF PHYSICAL KEY SEQUENCES; "COMPLEMENTARY NATURE" OF CERTAIN SCALES AND IMPLIED "PHYSICAL KEY GEOMETRY"; VIEWING CERTAIN "PHYSICAL KEY GEOMETRIES" AS RESULT OF REFLECTION AGAINST THE AXIS D-^{G}A.

• Keys, Scales, Circles

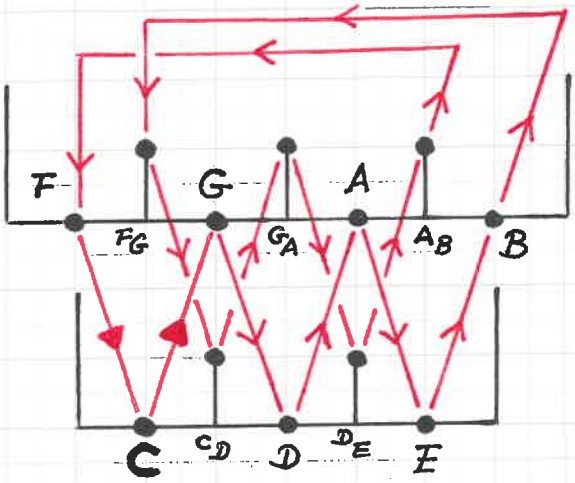
12/11/2024

• Abstract representation:



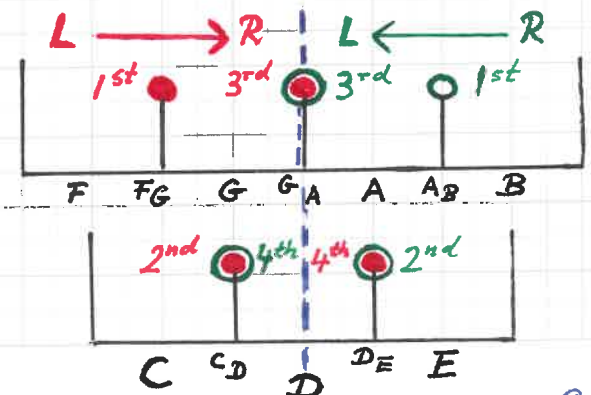


The figures above show the concept of the alternative block arrangement. Left: Visualization of physical key layout with note names and possible indexing / numbering. Right: More abstract visualization of layout, emphasizing the 2212221 halftone-jump sequence of C major.



Circle of 5ths.

The left figure represents the circle-of-5th order of notes (C-G-...$^{F}_{G}$-$^{C}_{D}$-...-F-C), shown by the red polygon with directed edges. The positions (•) for notes F, C, G, D, A, E, B "move from left to right"; the same holds for $^{F}_{G}$, $^{C}_{D}$, $^{G}_{A}$, $^{D}_{E}$, $^{A}_{B}$.



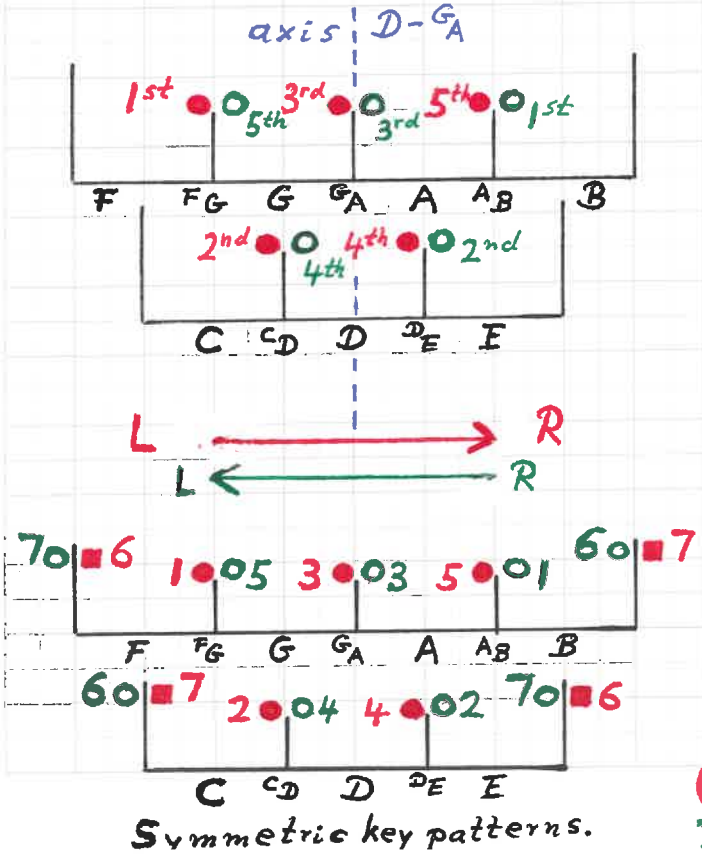
Symmetry axis of patterns: D-$^{G}_{A}$.

The left figure visualizes the order of introduction of '#'s (•) and 'b's (○), as implied by the key sequences C-G-D-A-E and C-F-B^{b}-E^{b}-A^{b}, respectively. The associated "directions" are left-to-right (L→R) and right-to-left (L←R).

• Keys, Scales, Circles 12/12/2024

Thus, the most important facts about this visualization are: (i) The L→R introduction of '#'s (and the corresponding physical keys) is "complemented" by the L←R introduction of 'b's (and the corresponding physical keys). (ii) The physical key patterns/shapes resulting after the introduction of 1 (2,3,4,5) '#' and 1 (2,3,4,5) 'b', respectively, are axis-symmetric relative to the D-G_A axis. THESE FACTS ARE VERY HELPFUL FOR MEMORIZING SCALE PROPERTIES ON THE PHYSICAL KEYBOARD.

First, we consider the "key patterns" that result from the introduction of 1 '#' and 1 'b'; 2 '#'s and 2 'b's; 3 '#'s and 3 'b's; 4 '#'s and 4 'b's; and 5 '#'s and 5 'b's. In other words, we are interested in the sequence of patterns resulting from the circle-of-5th keys G→D→A→E→B and F→B^b→E^b→A^b→D^b. The patterns are symmetric:



Symmetric key patterns.

The left figure visualizes the symmetry of the "black-key patterns" — being symmetric with respect to the D-G_A axis and evolving from left to right (L→R) and from right to left (L←R) when introducing '#'s and 'b's, respectively. The left figure captures the symmetric keyboard patterns for the introductions of 7 '#'s (F_G, C_D, G_A, D_E, A_B, $E^{\#}=F$, $B^{\#}=C$) and 7 'b's (A_B, D_E, G_A, C_D, F_G, $C^b=B$, $F^b=E$).

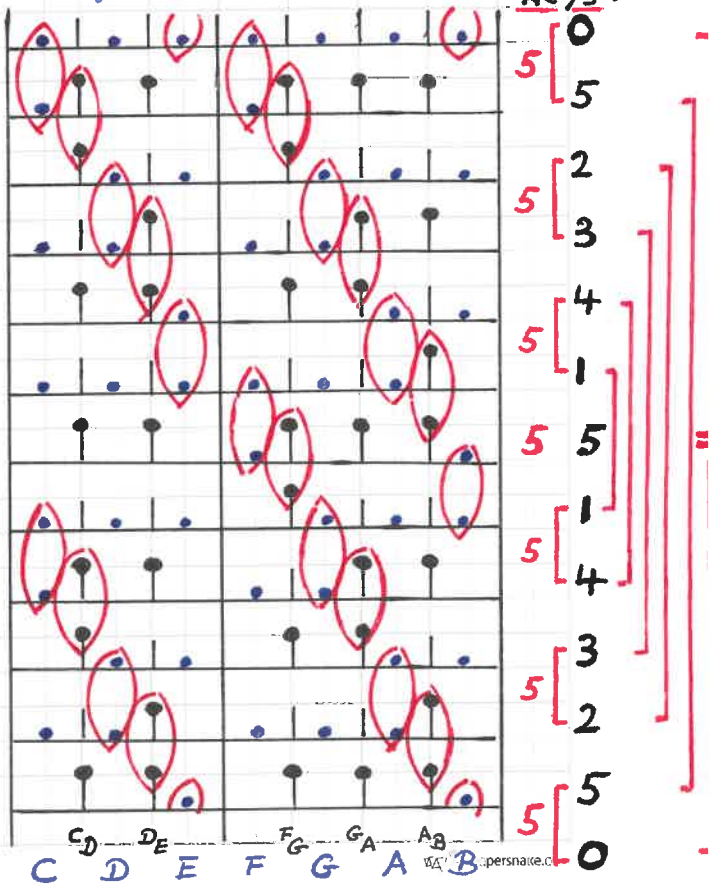
• Keys, Scales, Circles 12/12/2024

Second, we consider the concept of "complementary-keys, scales and keyboard patterns." IN ORDER TO DETERMINE POTENTIALLY MEANINGFUL AND HELPFUL "COMPLEMENTARY PATTERNS," IT SEEMS REASONABLE TO CONSIDER THE "BEHAVIOR OF ALL POSSIBLE PAIRS," E. G., PAIRS OF KEYS OR SCALES. Again, the objective is the identification of patterns useful for memorizing and playing. It is not obvious how to define two "complementary keys" or two (associated) "complementary keyboard patterns" (of the associated scale!). For example, one can consider the 7 notes of a specific key/scale as a subset of the 12 notes of the chromatic scale; thus, one can use a key/scale with notes that include the 5 "missing notes" — with respect to the "complete 12-note chromatic note set" — a complementary key/scale.

COMPLEMENTARY SCALE PAIRS

C	C	D	E	F	G	A	B
$D^b = C^\#$	C_D	D_E	F	F_G	G_A	A_B	C
D	D	E	F_G	G	A	B	C_D
$E^b = D^\#$	D_E	F	G	G_A	A_B	C	D
$F^b = E$	E	F_G	G_A	A	B	C_D	D_E
F	F	G	A	A_B	C	D	E
$G^b = F^\#$	F_G	G_A	A_B	B	C_D	D_E	F
G	G	A	B	C	D	E	F_G
$A^b = G^\#$	G_A	A_B	C	C_D	D_E	F	G
A	A	B	C_D	D	E	F_G	G_A
$B^b = A^\#$	A_B	C	D	D_E	F	G	A
$C^b = B$	B	C_D	D_E	E	F_G	G_A	A_B

Twelve (major) scales.



Scale keyboard patterns.

• Keys, Scales, Circles

12/13/2024

THE BOTTOM-RIGHT FIGURE ON THE PREVIOUS PAGE IS AN ABSTRACT VISUALIZATION OF THE KEYBOARD PATTERNS OF THE 12 (24!) SCALES OF INTEREST. PAIRS OF NEIGHBOR PATTERNS (IN TWO CONSECUTIVE ROWS) BELONG TO TWO COMPLEMENTARY KEY/SCALE PAIRS (BOTTOM LEFT). THE KEYBOARD PATTERNS MUST BE "TRANSLATED" INTO "PATTERN SEQUENCE RULES" THAT CAN BE MEMORIZED FOR PLAYING.

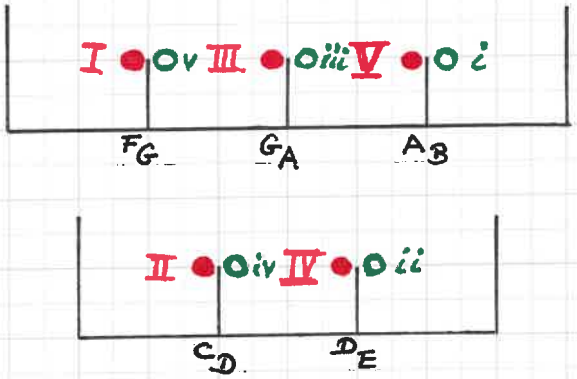
Instead of using the key/scale order induced by the chromatic scale, one can also use the circle-of-5th order to define complementary keys/scales. For example, one can consider pairs of keys and their numbers of associated '#'s and 'b's, written as tuples (N#, Nb) in a table:

b \ #	$_0$C	$_1$G	$_2$D	$_3$A	$_4$E	$_5$B	$_5$F#	$_5$C#
$_0$C	0,0	1,0	2,0	3,0	4,0	5,0	6,0	7,0
$_1$F	0,1	1,1	2,1	3,1	4,1	5,1	6,1	
$_2$B^b	0,2	1,2	2,2	3,2	4,2	5,2		
$_3$E^b	0,3	1,3	2,3	3,3	4,3			
$_4$A^b	0,4	1,4	2,4	3,4	4,4			
$_5$D^b	0,5	1,5	2,5			5,5		
$_5$G^b	0,6	1,6					...	
$_5$C^b	0,7							

The '#' keys are listed in the columns and have between 0 and 7 '#'s, with between 0 and 5 implied BLACK keys (indicated by the lower-left index). The 'b' keys are listed in the rows and have between 0 and 7 'b's, with between 0 and 5 implied BLACK keys (lower-left index). The table's diagonal shows tuples for which N# = Nb. These were already viewed as belonging to complementary keys — since their associated '#' and 'b' keyboard patterns are axis-symmetric relative to the D-G^A axis. DO TUPLES REPRESENT COMPLEMENTARY PATTERNS?

• Keys, Scales, Circles 12/14/2024

• A MORE GENERAL WAY TO VIEW "COMPLEMENTARY" — I.E., NOT ONLY CONSIDERING KEYS WITH AXIS-SYMMETRIC KEYBOARD PATTERNS — IS BASED ON KEYS(PAIRS!) THAT SATISFY THE CONDITION N#+Nb= const. For example, the key tuples satisfying N#+Nb=5 are underlined in the table on the previous page. Six key pairs satisfy this condition, where their combined total number of '#'s and 'b's (5) implies complementary BLACK key keyboard patterns that involve all 5 BLACK keys. The specific key pairs are: (B, C), (E, F), (A, B^b), (D, E^b), (G, A^b), (C, D^b). The left figure shows, in a combined manner, the resulting six BLACK key combinations that all involve all black keys: (I, II, III, IV, V), (I, II, III, IV, i), (I, II, III, i, ii), (I, II, i, ii, iii), (I, i, ii, iii, iv), (i, ii, iii, iv, v). Analogously, one can also use the key conditions N# + Nb = const, with const = 1, 2, 3, 4, for the other tuples in the table that imply the use of only a subset of the five black keys when combining the '#'s and 'b' of a key pair. Similarly, one can also consider the conditions N# + Nb = const, with const = 6, 7. In these cases, certain BLACK keys are "used twice" — since they are implied by both keys of a key pair.

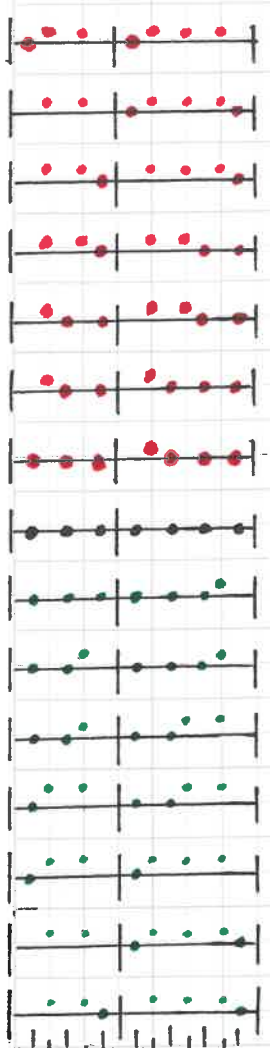


• Yet another possibility to define "complementary keys/scales" can use the number of identical or different keyboard keys of pairs of keys as basis (degree of similarity/dissimilarity).

• Keys, Scales, Circles

12/14/2024

			Cb	Gb	Db	Ab	Eb	Bb	F	C	G	D	A	E	B	F#	C#
a#	C#	C#	5(2)	6(1)	7(0)	6(1)	5(4)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)
d#	F#	F#	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)
g#	B	B	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)
c#	E	E	6(1)	5(4)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)
f#	A	A	5(4)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)
b	D	D	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)
e	G	G	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)
a	C	C	2(5)	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)
d	F	F	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)
g	Bb	Bb	2(5)	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)
c	Eb	Eb	3(4)	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(4)
f	Ab	Ab	4(3)	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(2)	6(1)
bb	Db	Db	5(2)	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(4)	6(1)	7(0)
eb	Gb	Gb	6(1)	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(4)	6(1)	7(0)	6(1)
ab	Cb	Cb	7(0)	6(1)	5(2)	4(3)	3(4)	2(5)	2(5)	2(5)	3(4)	4(3)	5(4)	6(1)	7(0)	6(1)	5(2)



Parallel minor keys

Major keys — keys played on keyboard (shown in 5-note and 7-note blocks)

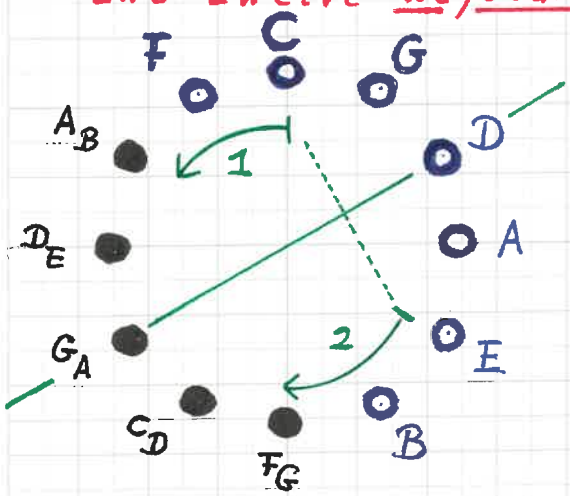
Table of pairs of major keys, listing the numbers of identical and different keys used for playing the two scales on the keyboard. The entry m(n) means that the two key pair implies m identical keys on the keyboard, while n = 7-m keys on the keyboard are different.

This symmetric table applies to all major and (parallel) minor keys. Underlined entries for key pairs high-light key pairs "very close to each other on the keyboard" (7(0), 6(1)) and pairs "far away from each other on the keyboard" (2(5)). THUS, ONE COULD VIEW THE "2(5)-pairs" AS KEY PAIRS THAT ARE "MAXIMALLY COMPLEMENTARY" TO EACH OTHER. For example, the pairs C-B, C-F# and C-C# have maximally different patterns, i.e., are complementary.

• Keys, Scales, Circles 12/15/2024

Based on this perspective, one could view 2(5)-pairs as pairs for which a "nearly complementary pattern of keyboard keys" must be memorized: ONE ONLY MUST MEMORIZE THE 2 KEYBOARD KEYS THAT ARE COMMON AND 5 KEYBOARD KEYS (OF ONE OF THE TWO SCALES) THAT ARE NOT IN COMMON (SINCE THE 5 KEYBOARD KEYS OF "THE SECOND SCALE" ARE COMPLEMENTARY).

• Of course, one can characterize "complementary" or "symmetric" behavior of keys, scales and keyboard patterns/shapes in several other possible ways. Generally, one can use set theory to define "complementary" behavior; and a geometrical viewpoint can be employed to define "symmetric" behavior. For example, one can observe these characteristics when using the circle-of-5th order to represent the twelve keyboard keys. The left figure does NOT represent the keys of the circle of 5ths; it simply uses the same circle arrangement to visualize the 12 keyboard keys of one octave, where the white keys (F, C, G, D, A, E, B) define the 7-note sequence and the black keys (F_G, C_D, G_A, D_E, A_B) define the 5-note sequence. One can use this layout of keyboard keys to "see" two "complementary and symmetric note sequences":



1: C F A_B D_E G_A C_D F_G B E A D G
2: E B F_G C_D G_A D_E A_B F C G D A

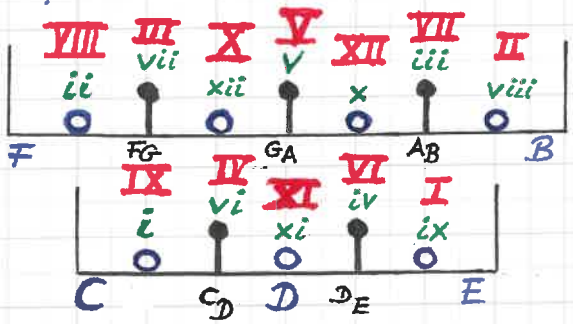
Both sequences start with 2 white keys, followed by 5 black keys, followed by 5 white keys.

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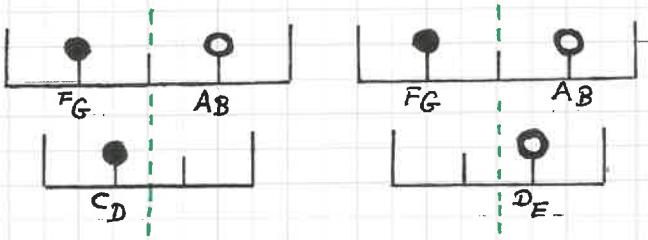
The "complementary and symmetric" nature of these two sequences of keyboard keys becomes evident when using the combined 5-key-and-7-key layout of the 12 notes/keys of the octave:



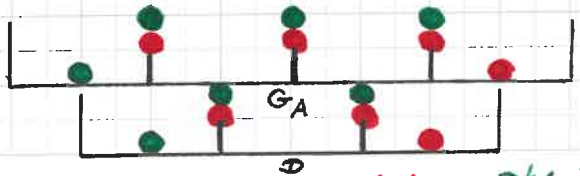
Two 12-tone key sequences that are symmetric relative to the axis D-GA.

This visualization that playing the first k ($1 \le k \le 12$) keys of the two sequences produces two symmetric "key pattern sequences": i, ii, iii, ..., k and I, II, III, ..., K.

• Specifically, one can consider the case of two keys (key signatures), where the first key has m associated '#'s and the second key has n associated 'b's. By defining $m+n=\text{const}=c$, one can also establish a "complementary key pair," where the first key has n '#'s and the second has m 'b's. For example, given the initial key pair D maj. and F maj. ($m=2$, $n=1$), the second "complementary pair" is G maj. and B^b maj. Thus, these pairs satisfy $m+n=3$. The pairs' symmetric keyboard patterns can be visualized via the 5-note and 7-note block representation:



Black keys of D(•), F(○). Black keys of G(•), B^b(○).



All 7 keys used by B(•) and D^b(•).

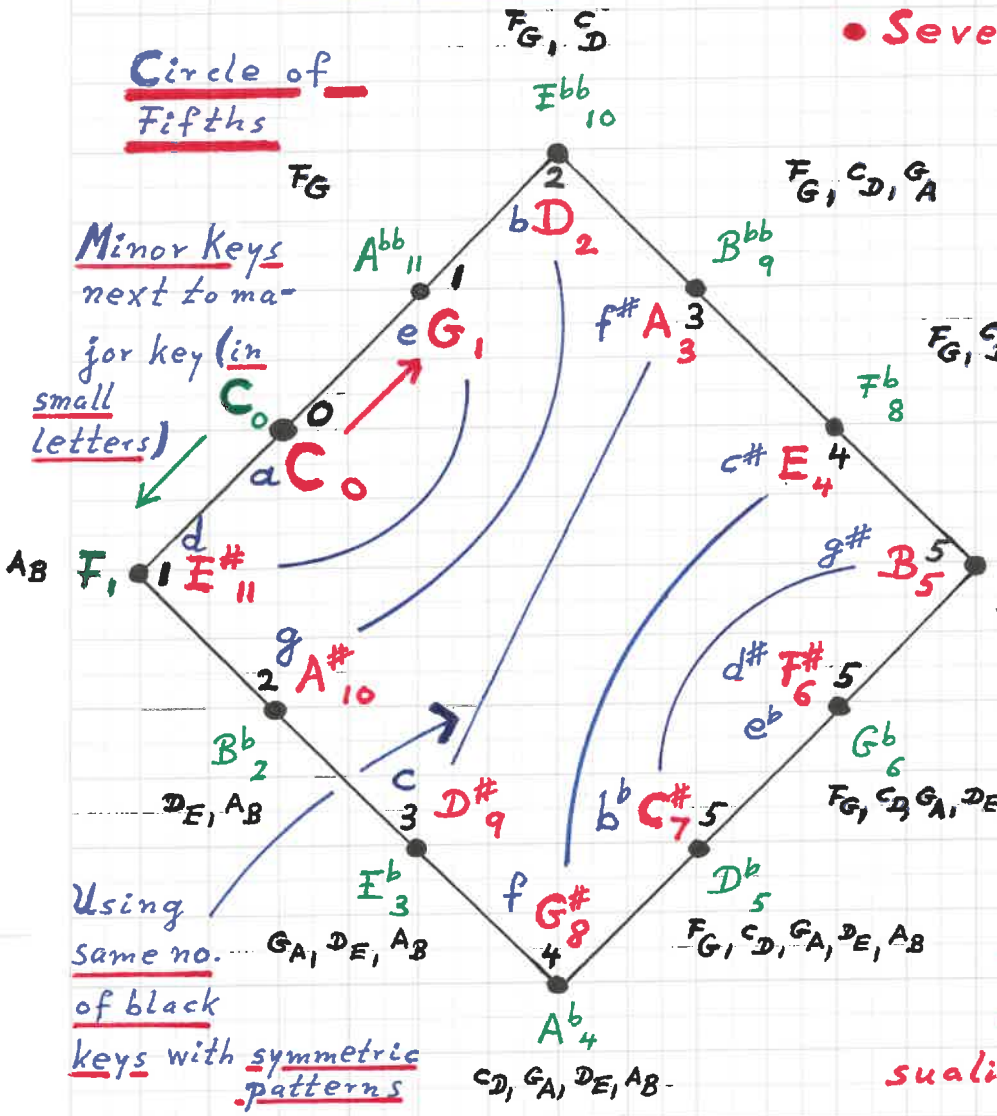
The left figure shows the black keys to be used for the pairs (D, F) and (G, B^b). The resulting keyboard patterns are symmetric relative to axis D-GA.

The left figure shows the 7-key keyboard key-set patterns for B maj. and D^b maj. They are symmetric.

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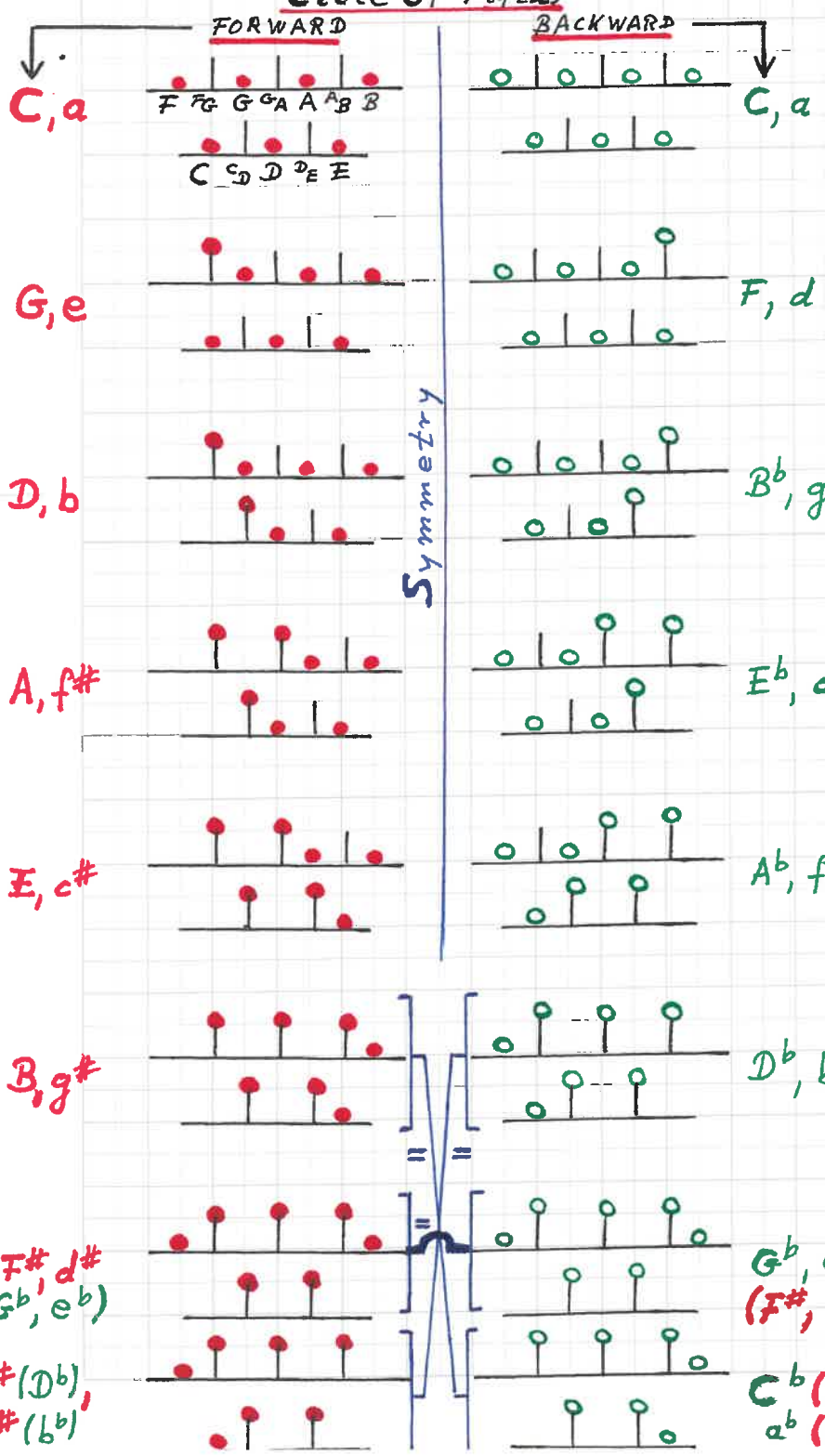
• Several facts that are relevant for the identification of relationships between keys (key signatures) and resulting symmetric and complementary characteristics can be visualized by the circle of 5th's.

The figure above includes all 12 keys obtained by moving forward by 5th's and all 12 keys obtained by moving backward by 5th's. Next to a '#-key' the number of '#'s is shown; next to a 'b-key' the number of 'b's is shown. At every position '•', these two numbers always add up to 12 (or 0). Next to each '•', the number of black keyboard keys implied by the associated pair of keys (key signatures) is given; this number is between 0 and 5, of course. The blue lines in the interior of the square connect keys (key signatures) with keyboard key patterns that are symmetric — when considering their entire scales or only their implied black keys.

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The visualization of the 12 halftones from C to B via a 5-note and 7-note block and considering the described concepts of "complementary and symmetric" patterns of scales (note sets) played on the keyboard makes it possible to group the 12 major and 12 minor keys:



The figure on the left shows the 7 keys on the keyboard that must be played for the 12 major and 12 minor scales. The IMPORTANT INSIGHT IS THE FACT THAT THE PAIRS OF LEFT ("FORWARD") AND RIGHT ("BACKWARD") PATTERNS OF 7 KEYBOARD KEYS ARE SYMMETRIC. THUS, ONE MUST MEMORIZE ONLY THE SETS / KEYS FOR MAJOR KEYS C, G, D, A, E, B, F♯.

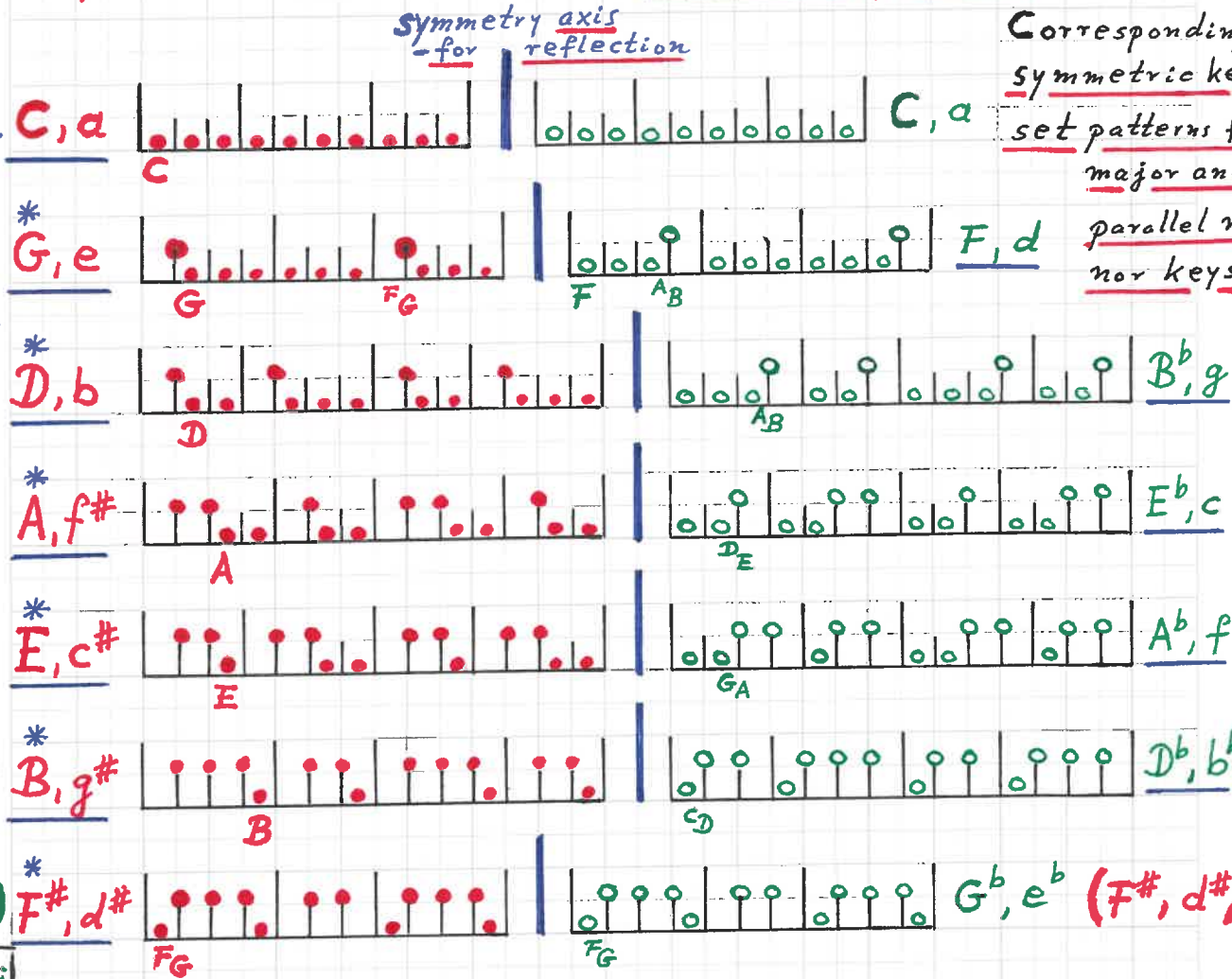
C MAJOR INVOLVES WHITE KEYS; THUS, SIX SETS MUST BE MEMORIZED. FURTHER, A MAJOR KEY AND ITS PARALLEL MINOR KEY SHARE THE SAME SET OF 7 KEYBOARD KEYS

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Instead of using the visualization with the 5-note and 7-note blocks to represent symmetric patterns of note/key sets, one can also see the symmetric patterns on the actual "linear keyboard":

The 24 keys are underlined. The SIX patterns to be memorized are marked with an '*'.



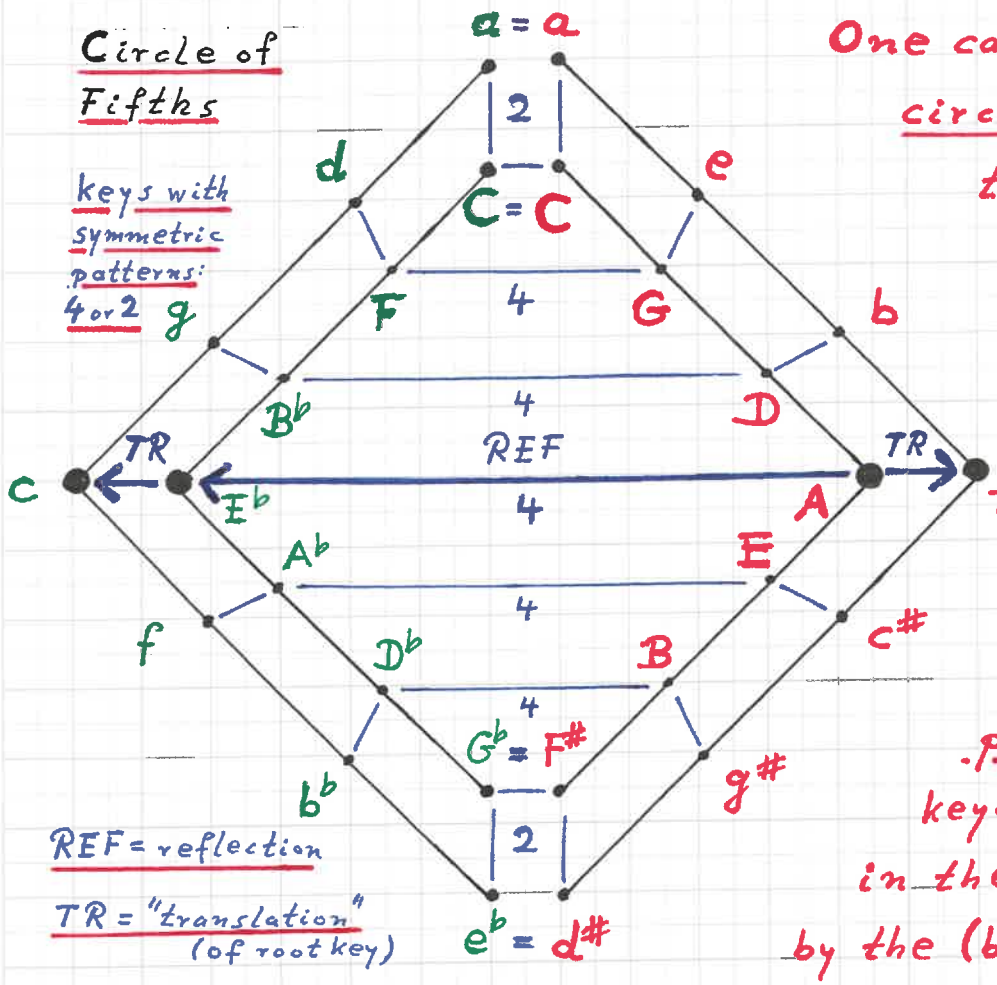
maj.	min.	refl. maj.	refl. min.
C	a	C	a
G	e	F	d
D	b	B♭	g
A	f♯	E♭	c
E	c♯	A♭	f
B	g♯	D♭	b♭
F♯	d♯	G♭	e♭

Table with rows listing keyswith symme-

These visualizations of pairs of note/key sets show symmetry: The sets on the right-hand side are obtained by "reflecting" the sets on the left-hand side using the symmetry axis as reflection axis. Alternatively, one can understand a pattern on the right-hand side as the result of "traversing" the pattern on the left-hand side from right to left. Again, ONE MUST MEMORIZE ONLY SIX PATTERNS — G, D, A, E, B, F♯. Other key patterns are implied by reflection, and major and parallel minor keys have the same set patterns.

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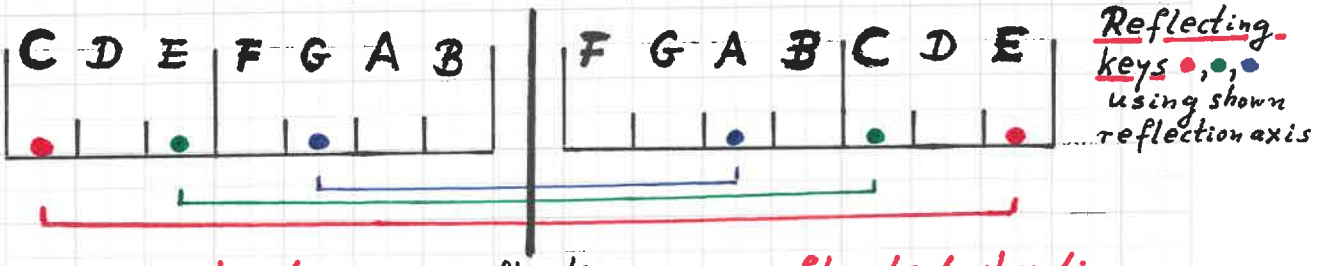


One can use the circle of fifts to represent/visualize the table on the previous page. The keys with symmetric patterns on the keyboard are shown in the circle of fifths by the (blue) edges that connect 4 keys. For example, remembering/knowing the scale for A major, one can "immediately play" the scales for E^b major, $f^\#$ minor and c minor: The scale for E^b major is obtained from the scale for A major by reflection; the scale for $f^\#$ minor is the scale of the parallel minor of A major, thus using the same keys as those used for A major — but using $f^\#$ as root key; the scale for c minor is the scale of the parallel minor of E^b major, thus using the same keys as those used for E^b major — but using c as root key. (This example is high-lighted in bold in the above figure.) The two special cases are shown at the top of the circle (C-a) and at the bottom ($(F^\#-d^\#)=(G^b-e^b)$). KNOWING THE CIRCLE OF FIFTHS AND THE SCALES FOR G, D, A, E, B, $F^\#$ SUFFICES.

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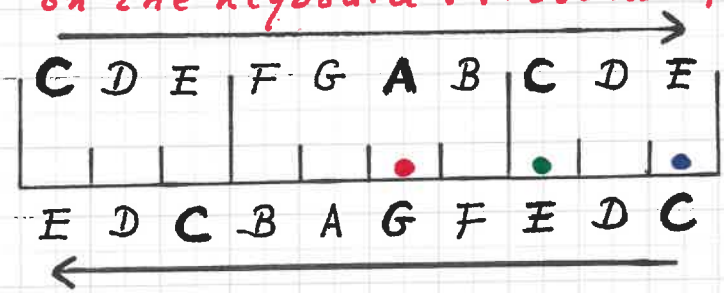
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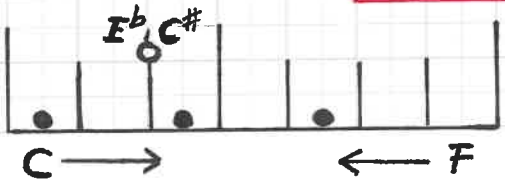
• ANOTHER INTERESTING POSSIBILITY FOR MEMORIZING "SHAPES AND PATTERNS" ON THE KEYBOARD IS THE USE OF REFLECTION: ONE CAN "INVERT" A KEY SEQUENCE BY "READING THE KEYBOARD FROM RIGHT TO LEFT." WHAT IS MEANT BY THIS STATEMENT? THE GENERAL PRINCIPLE IS DESCRIBED.



⇒ The chord CEG = Cmaj = C becomes the reflected chord ACE = Amin = a. ⇒ Major becomes minor.

• One can also think of the reflection operation as a reversal of the naming / labeling of the keys on the keyboard: Instead of using the "standard" left-to-right naming sequence CDEFGAB, one can consider the right-to-left naming sequence EDCBAGF for the same seven keys. The figure shows how these two naming / labeling options are applied to the keyboard. The figure shows how the "given chord" ACE = Amin = a becomes the "new chord" CEG = Cmaj = C. ⇒ Minor becomes major.





→ CEG, CE♭G The left figure

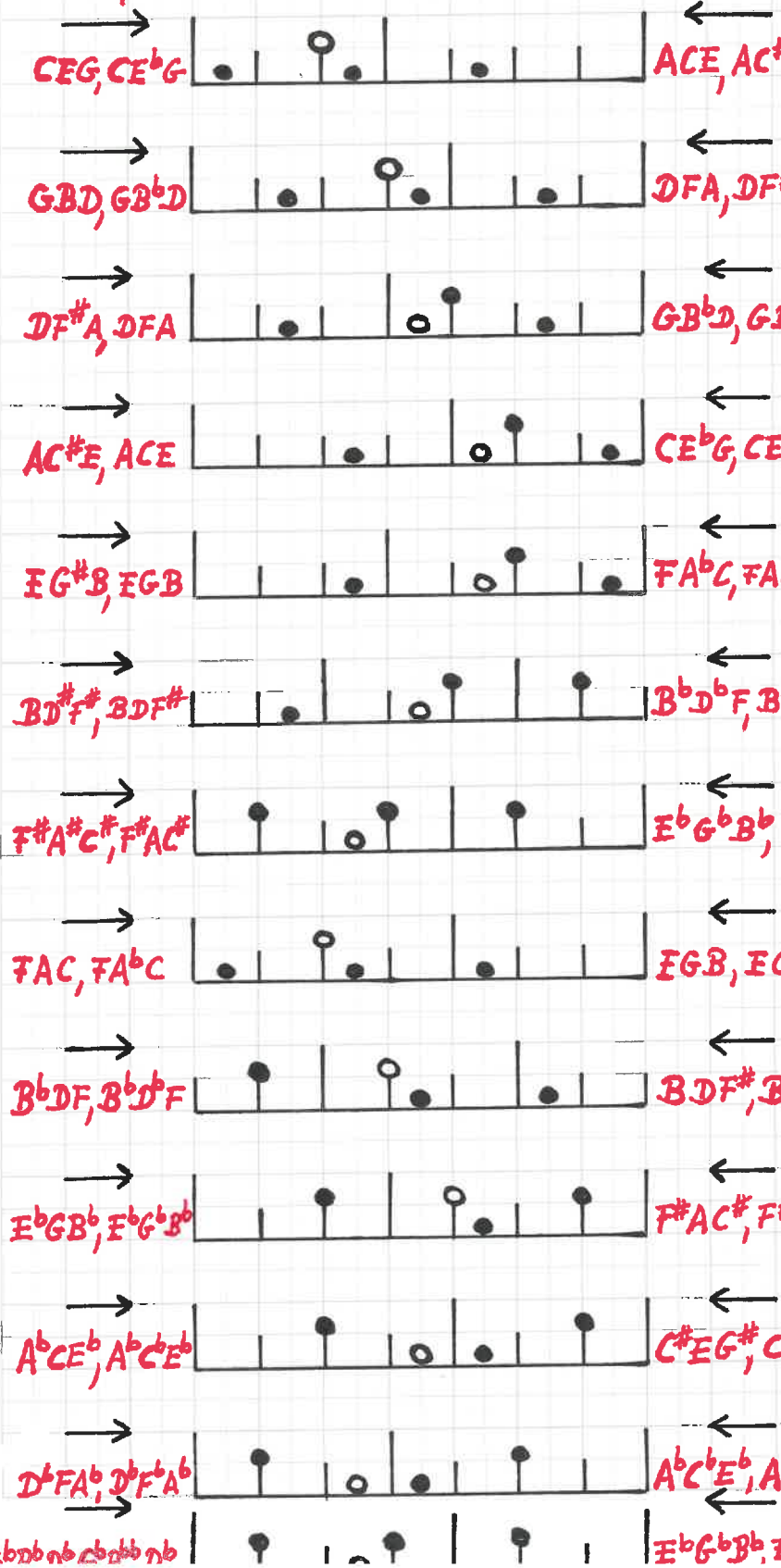
← ACE, AC♯E shows the chords CEG and CE♭G – mapped to ACE and AC♯E

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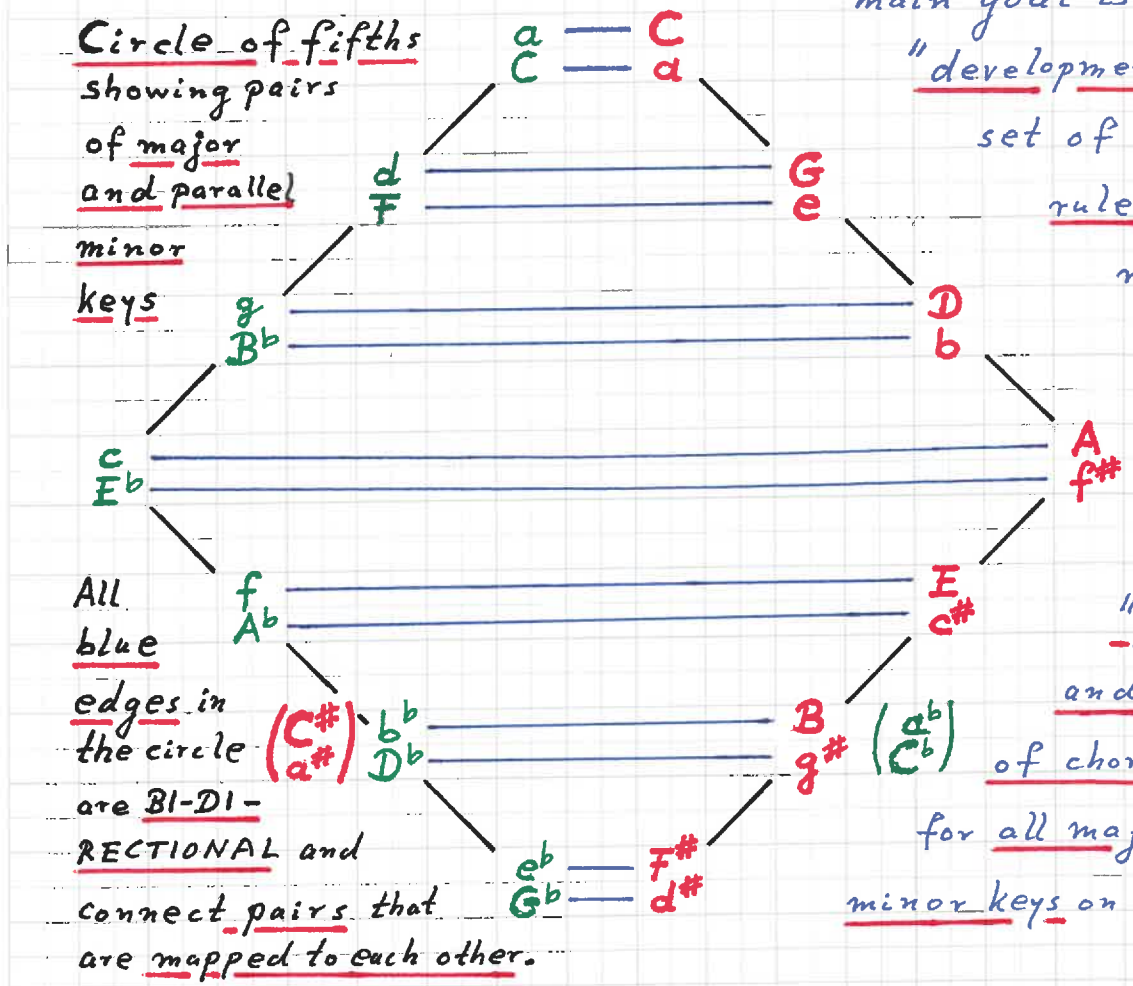
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In the following, the major and minor triads of all keys are mapped based on the principle described on the previous page. The circle-of-fifths order of keys is used to perform / list all mappings.



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It is possible – and helpful – to represent the "harmonic relationships" resulting from this mapping in the context of the circle of fifths. One also must keep in mind that the following (maj, min) pairs are equivalent: $(F^{\#}, d^{\#}) \triangleq (G^{b}, e^{b})$; $(C^{\#}, a^{\#}) \triangleq (D^{b}, b^{b})$; $(B, g^{\#}) \triangleq (C^{b}, a^{b})$. Once again, the main goal is the "development of a set of SIMPLE rules" for making it easier to memorize the "patterns and shapes" of chords/triads for all major and minor keys on the keyboard.



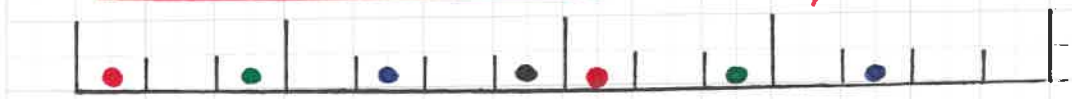
⇒ BY KNOWING AND MEMORIZING THE "MAPPING RELATIONSHIPS" INCLUDED IN THIS REPRESENTATION OF THE CIRCLE OF FIFTHS, THE TRIAD "SHAPES" OF CHORDS MAPPED TO EACH OTHER CAN BE PLAYED/DERIVED MORE EASILY.

→ C	C_D	D	D_E	E	F	F_G	G	G_A	A	A_B	B
← E	D_E	D	C_D	C	B	A_B	A	G_A	G	F_G	F

The left figure shows the principle. D–G_A remains as "axis."

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• In addition, one can use the described "reversal of the keyboard direction" for triad and 7th chord inversions. For example, we consider a C chord:



→ C D E F G A B C D E F G A B

← E D C B A G F E D C B A G F

7th chords:

C E G B	root pos.
E G B C	1st inv.
G B C E	2nd inv.
B C E G	3rd inv.

→ 7th's

C E G B
E G B C
G B C E
B C E G

← 7th's

F A C E	root pos.
A C E F	1st inv.
C E F A	2nd inv.
E F A C	3rd inv.

reverse order 7th chords:

A C E F	1st inv.
C E F A	2nd inv.
E F A C	3rd inv.
F A C E	root pos.

"Included triads":

→	←	
C E G	A C E	root pos.
E G C	C E A	1st inv.
G C E	E A C	2nd inv.
(C maj)	(A min)	

⇒ A given 7th chord implies 14 "patterns":

- 4 patterns for its root and 3 inversion forms
- 4 patterns for root and 3 inversion forms when using reverse keyboard order
- 3 patterns for its "included triad" in root and 2 inversion forms
- 3 patterns for root and 2 inversion forms for its "included triad" when using reverse keyboard order

Since there are 5 commonly used 7th chords for a specific root note, the number of "patterns" for a root note is 5×14 = 70. Since there are 12 possible root notes, the total number of "patterns" is 12×70 = 840.

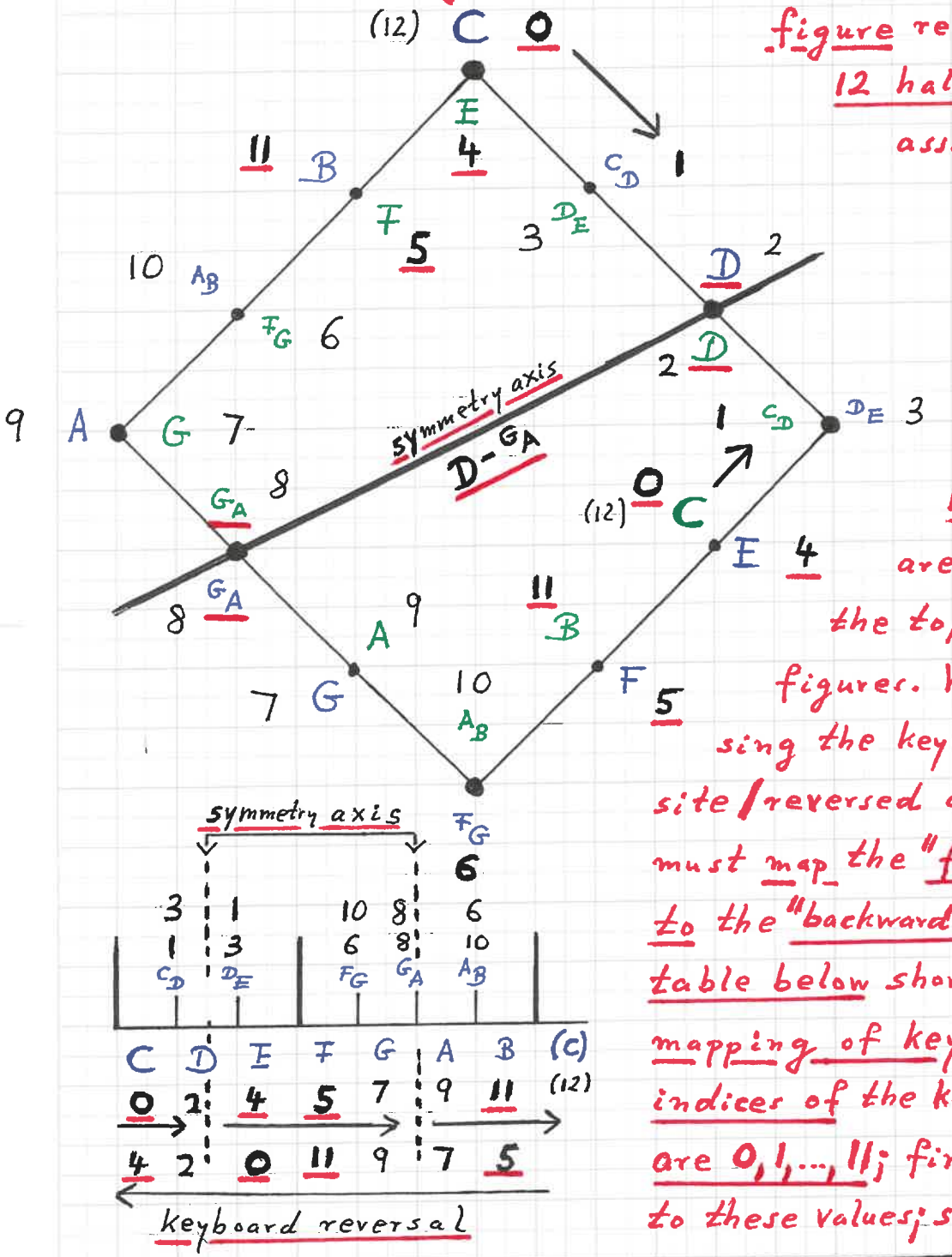
⇒ It is highly desirable to understand relationships between these "patterns" in order to simplify the process of memorizing and playing the "patterns." Thus, one must determine these relationships and derive "simple pattern rules" for playing.

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• One can also view the "reversal of the keyboard direction" from a purely geometrical / linear transformation / algebraic perspective. The left figure represents the 12 halftones and associated keys on the keyboard in an abstract way. Note names, keys, numbering / indexing are included in the top and bottom figurer. When "traversing the keyboard in opposite / reversed direction" one must map the "forward index" to the "backward index." The table below shows the needed mapping of key indices. The indicer of the keys for C, C_D, ..., B are 0, 1, ..., 11; first, one adds 16 to these values; second, one uses the operation "modulo 12" to obtain the "backward index."

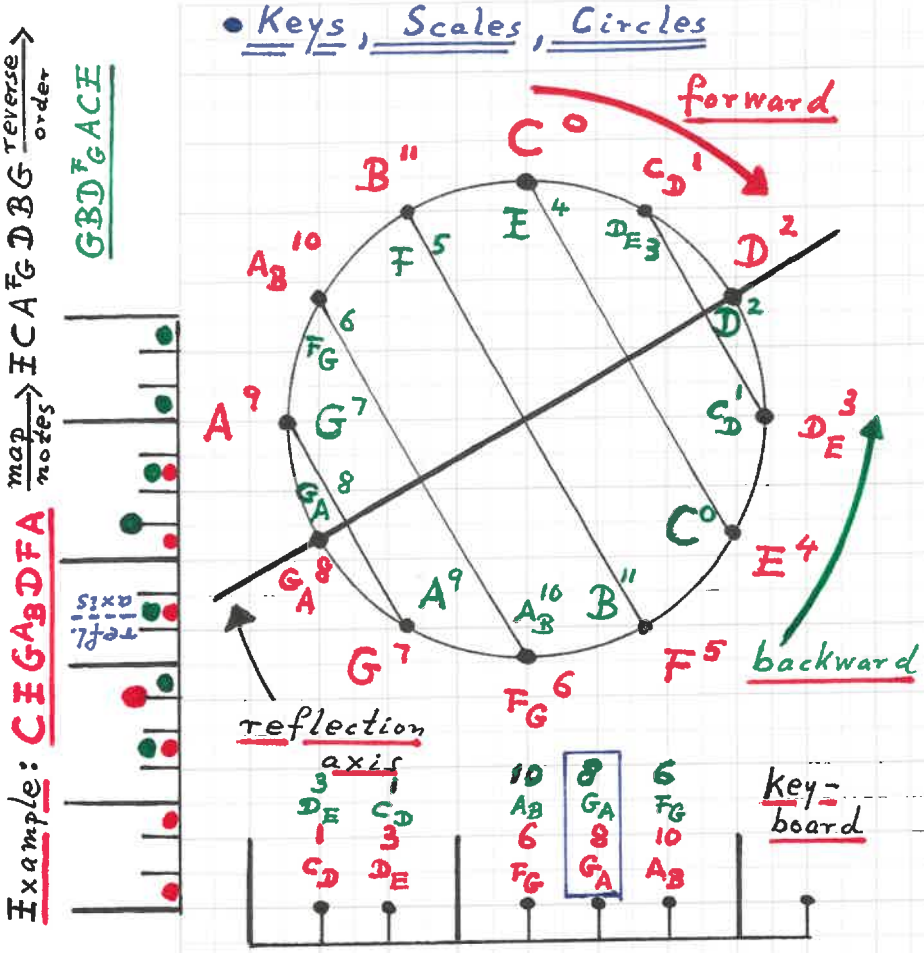


forward index i	0	1	2	3	4	5	6	7	8	9	10	11	12
$I = 16 - i^*$	16	15	14	13	12	11	10	9	8	7	6	5	4
backward ind. (I mod 12)	4	3	2	1	0	11	10	9	8	7	6	5	4

*One could also define I as $I = 4 - i$.

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mapping note names	C	D	E	F	G	A	B	C	← index i
	0	2	4	5	7	9	11	12	mapping key indices
	E	D	C	B	A	G	F	E	
index j →	4	2	0	11	9	7	5	4	

Index mapping: $j = (16 - i) \bmod 12$

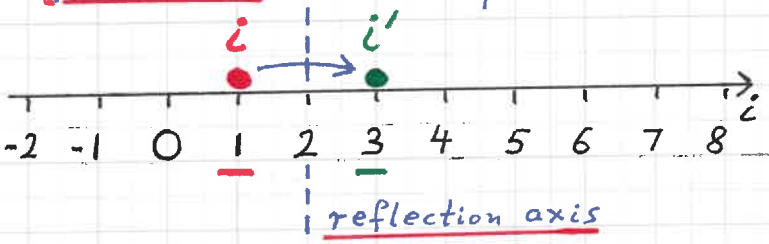
The left figure shows how the symmetry axis D-G_A can geometrically be viewed as a "reflection axis" that maps "key indices and note names from one side to the other side" of the axis. The symmetry of the arrangement of the physical keys of the keyboard leads to the index-mapping formula provided in the left figure. Since we have established a correspondence between notes/keys and index values for the FORWARD and BACKWARD directions, one can "invert/reverse/reflect" any given sequence of notes/chord by the defined mapping of key indices. For example, a given index tuple $(i_1, \ldots, i_n)$ is mapped to the tuple $(j_1, \ldots, j_n) = ((16-i_1) \bmod 12, \ldots, (16-i_n) \bmod 12)$. Considering the triad CEG, the triad's index tuple is (0,4,7). The mapping defines the new tuple $(16 \bmod 12, 12 \bmod 12, 9 \bmod 12) = (4, 0, 9)$. THE ORDER OF THIS TUPLE'S COMPONENTS MUST BE REVERSED, LEADING TO (9,0,4) ≙ ACE.

$(i_1, \ldots, i_n) \to (j_n, \ldots, j_1) = ((16-i_n) \bmod 12, \ldots, (16-i_1) \bmod 12)$

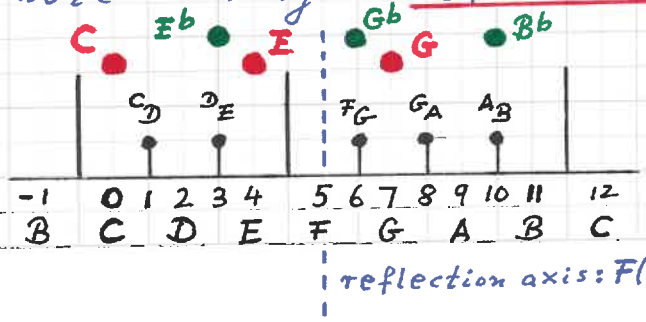
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• It is possible to understand and perform reflections in a general one-dimensional "line setting": Since one can assign a specific integer value to the keys (and notes), one can define a reflection of a key/note, chord or sequence of keys/notes relative to a chosen "reflection axis" (= a point / integer / key). The linear transformation is briefly described first. In the example shown in the left figure, the i-value used to define the reflection axis is i=2.



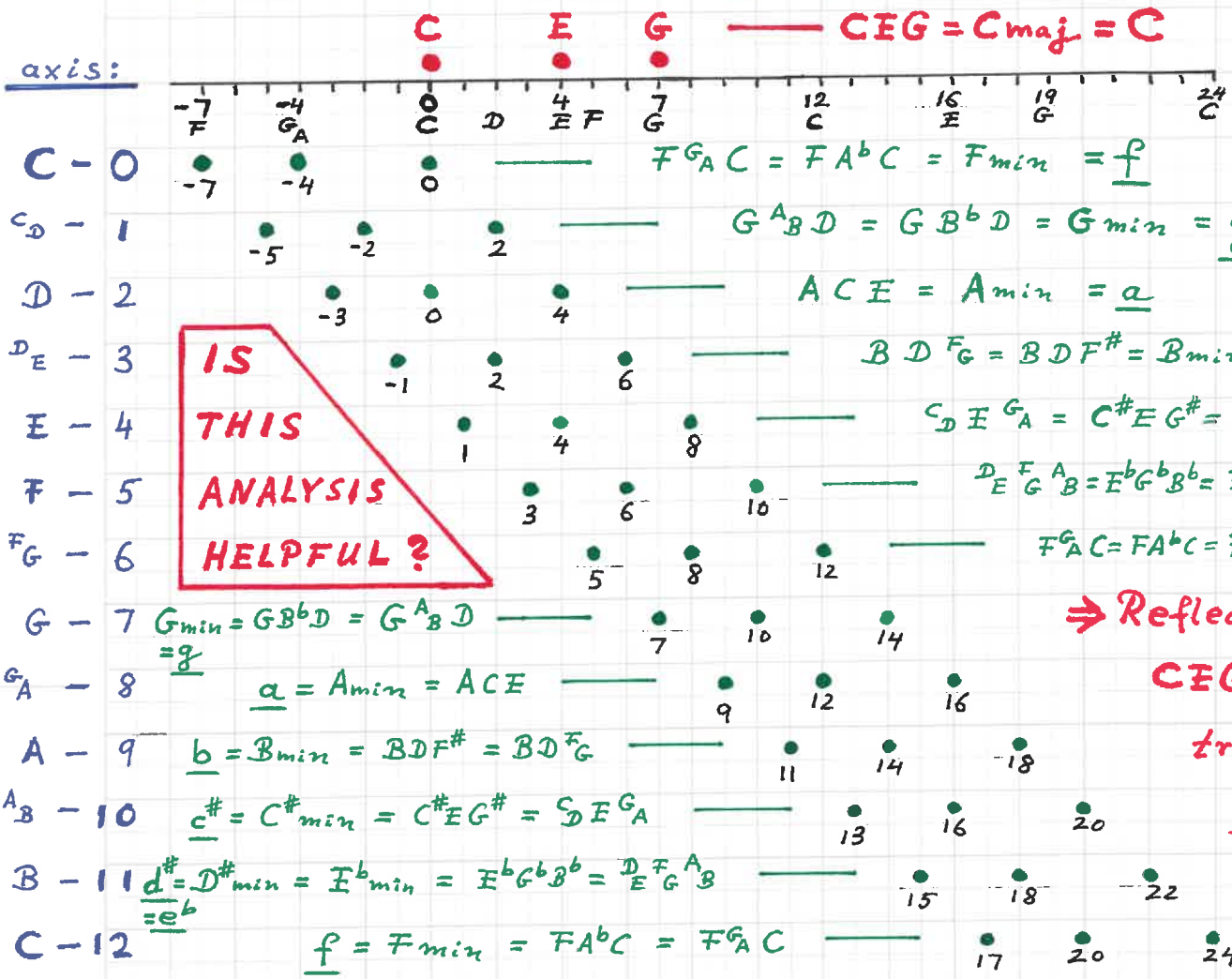
When reflecting the specific value i=1 relative to this axis, it is mapped to i'=3. One can understand the mapping as the concatenation of three "simple" steps: "Subtract 2 from the given i-value, then invert the sign, then add 2": $i' = (-(i-2)) + 2 = 4 - i$. Of course, on the keyboard such a reflection (of a specific key/note) relative to a reflection axis (a specified key/note) must be "seen and played immediately" — without performing "calculations." Obviously, one can use this transformation to reflect any group of keys/notes (chords!) relative to any key/note serving as reflection axis. Reflecting the Cmaj triad CEG using F(5) as reflection axis yields the triad $E^bG^bB^b$, i.e., the E^bmin triad, see left figure.



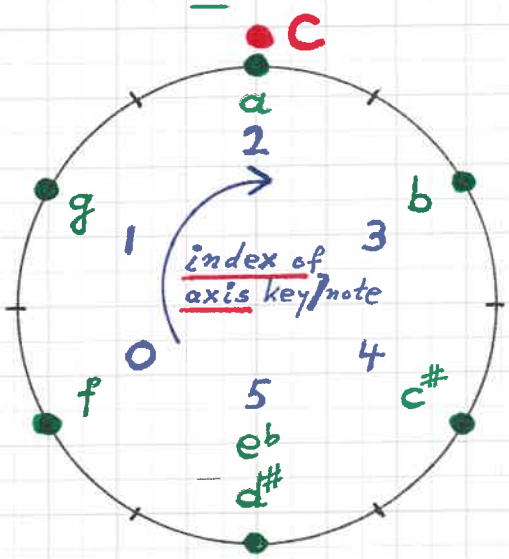
12/30/2024

• Keys, Scales, Circles

To obtain a better understanding of the reflection of chords, we reflect the Cmaj triad using several keys/notes to serve as reflection axis.



⇒ Reflecting the CEG major triad can only yield 6 minor triads!



Using circle-of-fifths illustration to show location of resulting triads.

When using the 12 halftones of the chromatic scale one obtains only 6 "reflection triads" of Cmaj (repeated in different octaves). The 6 distinct resulting triads are Fmin, Gmin, Amin, Bmin, C#min, D#min.

Thus, all triads are minor triads.

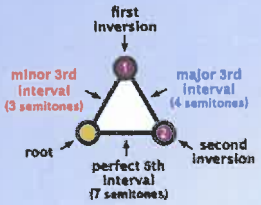
The circle-of-fifths layout nicely shows the "harmonic location" of the 6 minor triads, see left figure.

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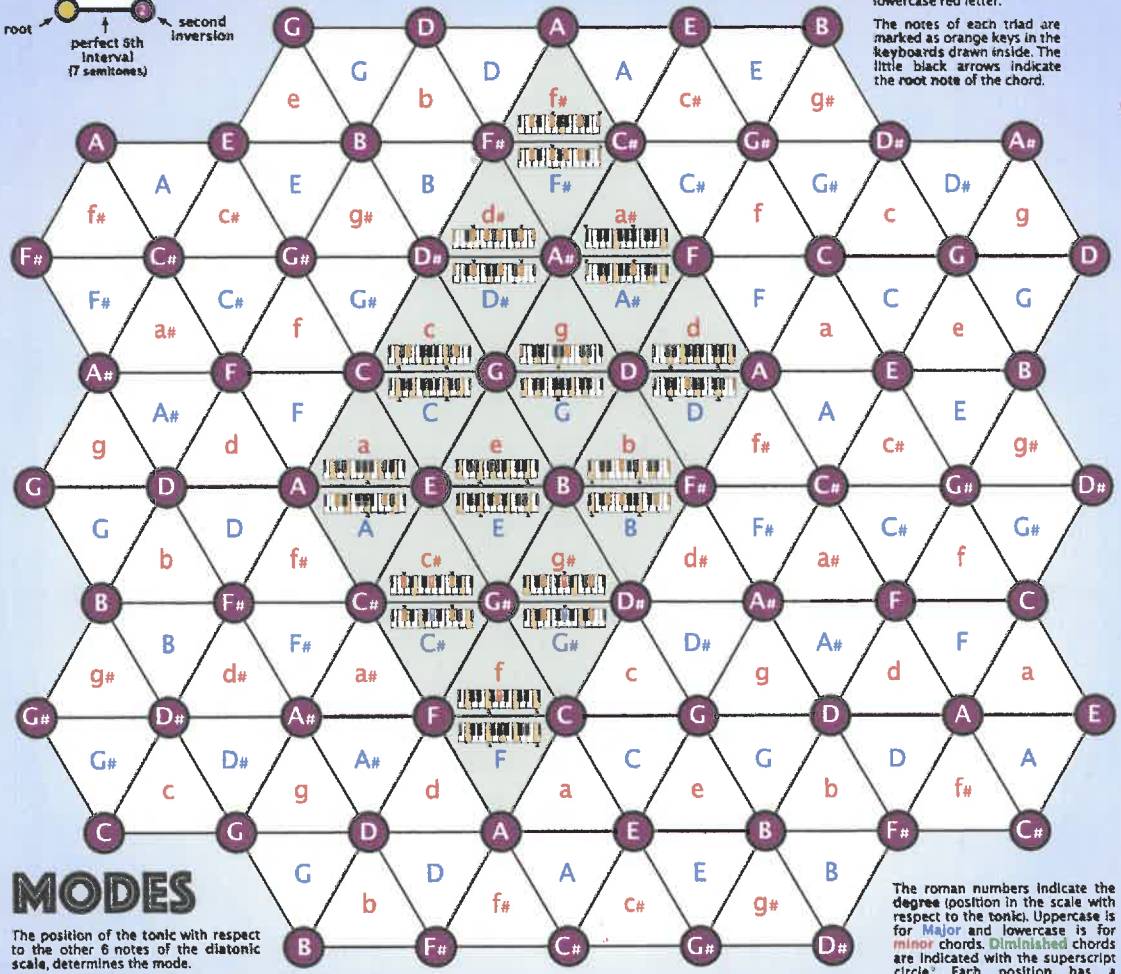
TONNETZ

In 1739 the mathematician Leonhard Euler discovered that the 12 notes of the western music system can be arranged in an hexagonal grid, the **Tonnetz**. The notes are at the nodes of the grid (purple disks) and the interval between adjacent notes in each direction is the same across the entire diagram. For example, there is a perfect 5th interval between C and G, as well as between G and D. The shadowed parallelogram represents the unit cell, the block of notes that is repeated infinitely in both directions of the plane.



Each triangle in the Tonnetz is a triad (group of 3 notes), corresponding to either a Major chord, labelled with an uppercase blue letter, or a minor chord, labelled with a lowercase red letter.

The notes of each triad are marked as orange keys in the keyboards drawn inside. The little black arrows indicate the root note of the chord.

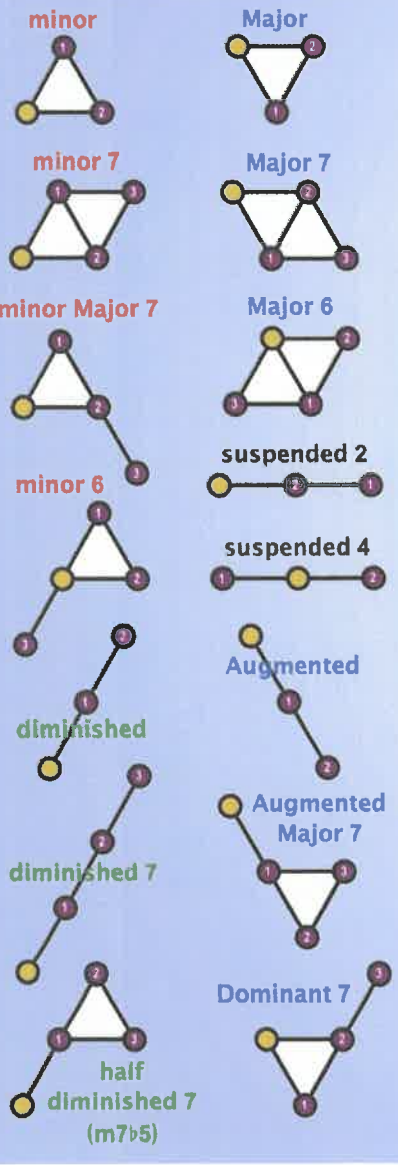


CHORDS

A **chord** is a set of 3 or more **notes** that sound harmonically when played together. Since chords are defined by the **interval** between its notes, in the **Tonnetz** the same type of chord (for example a Major chord) has always the same shape, regardless of where you put the **root note** (Cmaj, Dmaj, Emaj, ...).

The **root note** gives name to the chord and has the lowest pitch, unless the chord is **inverted**. The first inversion of a chord has the same notes, but the note at position 1 (see diagrams below) has the lowest pitch instead. Similarly, the second inversion gets note 2 as its lowest pitch note, etc.

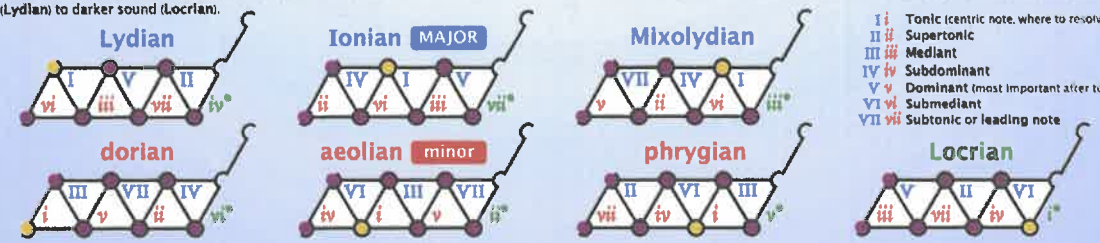
Inverted chords are indicated with the lowest pitch note as part of the name. For example: 2nd inversion of C minor = Cm/G.



MODES

The position of the tonic with respect to the other 6 notes of the diatonic scale, determines the mode.

Each of the 7 main modes of the diatonic scale has a different flavor in western music since the middle ages, from brighter sound (Lydian) to darker sound (Locrian).

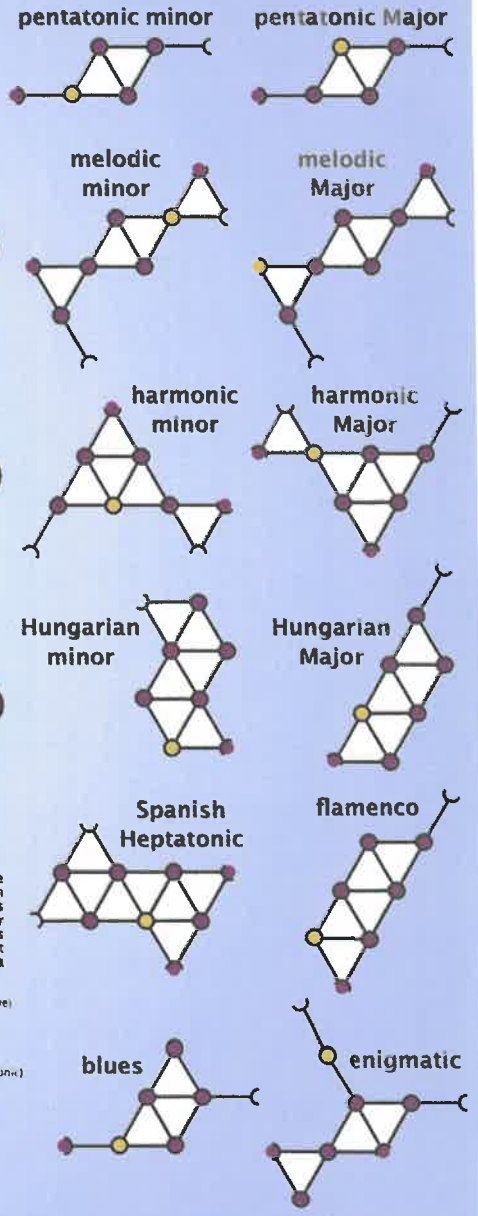


The roman numbers indicate the **degree** (position in the scale with respect to the tonic). Uppercase is for Major and lowercase is for minor chords. Diminished chords are indicated with the superscript circle°. Each position has a function in western tonal system:

- I i Tonic (centric note, where to resolve)
- II ii Supertonic
- III iii Mediant
- IV iv Subdominant
- V v Dominant (most important after tonic)
- VI vi Submediant
- VII vii Subtonic or leading note

SCALES

Any set of notes forms a **scale**. Scales are characterized by how many notes they have and by their relative positions. In the Tonnetz, each scale has a shape, and all the chords that fit into those shapes, are chords of the scale. The most common scale is the diatonic scale (see its 7 modes at the bottom). Here are some other common scales. Yellow disks mark the tonic note of the scale.



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Circle of Fifths (maj and min)

G^b	D^b	A^b	E^b	B^b	F	C	G	D	A	E	B	F#
e^b	b^b	f	c	g	d	a	e	b	f#	d^b	a^b	e^b

C# D# F# G# A#

C$_D$ D$_E$ F$_G$ G$_A$ A$_B$

D^b E^b G^b A^b B^b

12maj
12min

circle of m3rds

circle of 5ths

circle of M3rds

5th m3rd M3rd

circle of fifths

CE^{b}G minor

CEG major

notes of chromatic scale

"harmonic neighborhood" of triad at center

Five 7th chords:
CEGB
CEGA$_B$
D$_E$GA$_B$
D$_E$F$_G$A$_B$
D$_E$F$_G$A

Chords in the Tonnetz

-12-

0 1 2

012 root pos.
120 1st inv.
201 2nd inv.

3 M3rds = 1 octave
4 m3rds = 1 octave
12 5ths = 7 octaves

12/31/2024

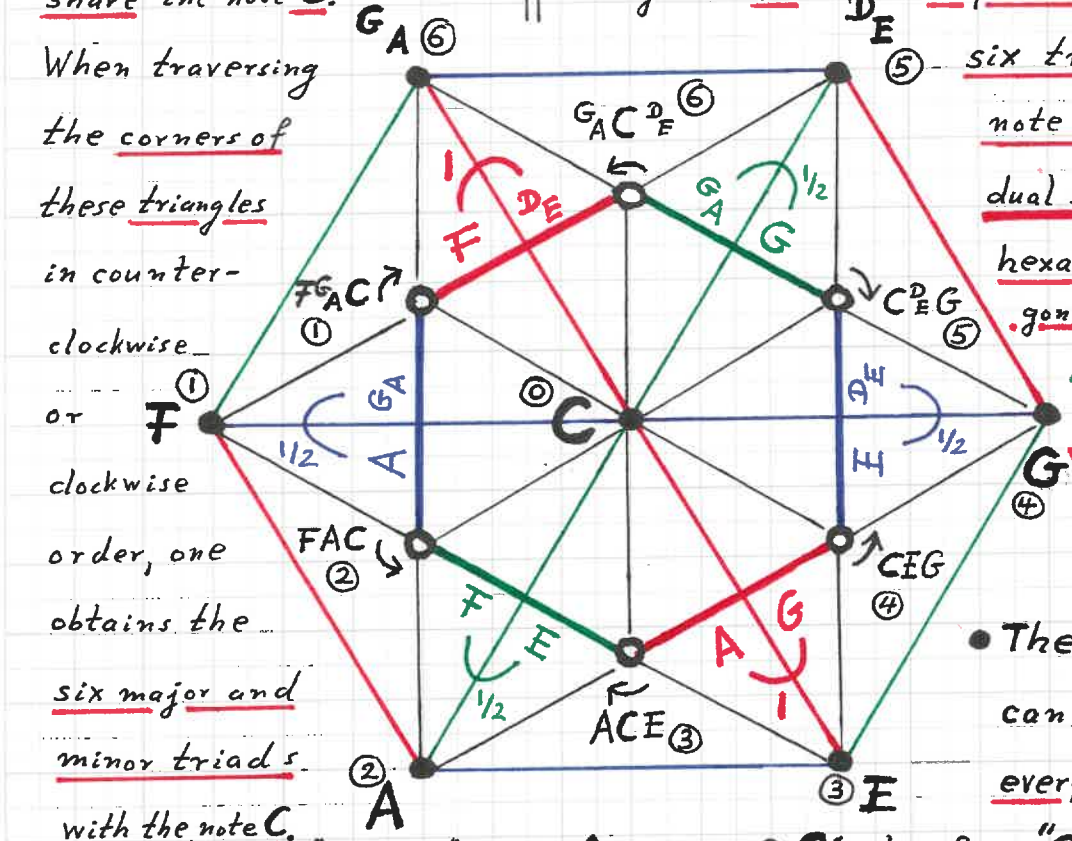
The 4×3 and larger 12×12 "structures" are periodic/cyclic in the horizontal/vertical directions.

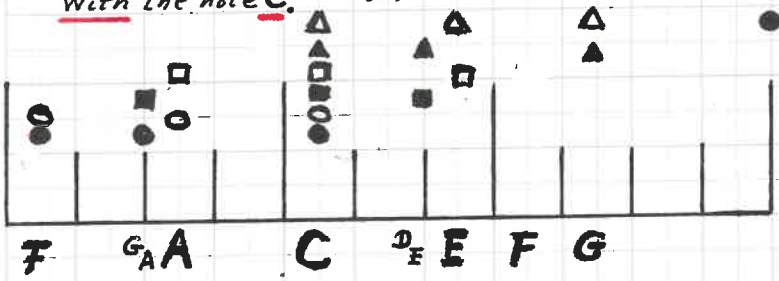
"Neighbors" of triads:
C maj: a, c, e min
a min: F, A, C maj

notes of chromatic scale

• Six Tonnetz-triangles share the note C. When traversing the corners of these triangles in counter-clockwise or clockwise order, one obtains the six major and minor triads with the note C.

• The Tonnetz structure (graph) is "designed" to represent/visualize the six triads that each note belongs to. The dual structure consists of hexagons.





• Circle of six "C-triads":

①	F	G_A	C	=	C	F	G_A
②	F	A	C	=	C	F A	1/2
③	A	C	E	=	C	A E	1/2
④	C	E	G	=	C	E G	1
⑤	C	D_E	G	=	C	D_E G	1/2
⑥	G_A	C	D_E	=	C	D_E G_A	1/2
①	F	G_A	C	=	C	F	G_A

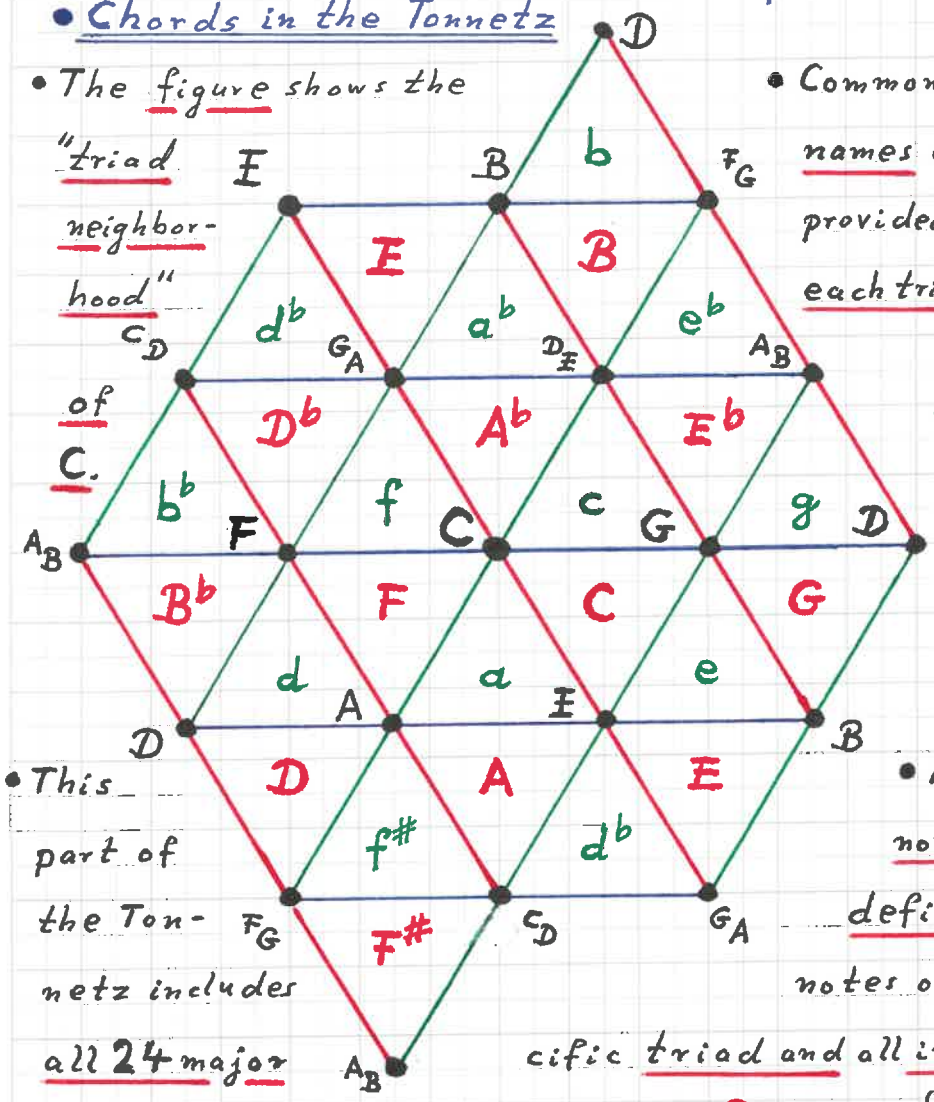
• The same structure can be defined for every note being at the center.

• The note C is part of six triads, shown in the keyboard sketch and the list of "C-triads" (right). The order of these triads is defined by the oriented "C-triad hexagon" in the figure above. By performing triad inversions of the "C-triads" such that C is the first note, one sees that a pair of subsequent/neighboring "C-triads" have have "total note distance" of either one (1/2) or two ($2 \cdot 1/2 = 1$) halftones. Thus, when "moving" forward or backward on the ordered "C-triad hexagon," the overall "smoothest C-triad movement" is guaranteed – ideal for modulation.

11/4/2025

• Chords in the Tonnetz

• The figure shows the "triad neighborhood" of C.



• This part of the Tonnetz includes all 24 major and minor triads.

• Common triad names are provided in each triangle.

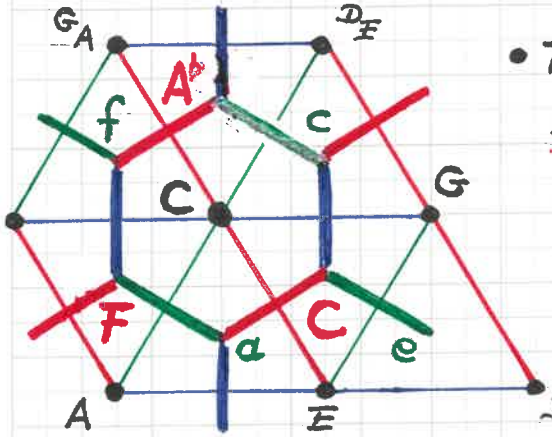
• A triad note set defines the notes of a specific triad and all inversions.

• Triad note sets:

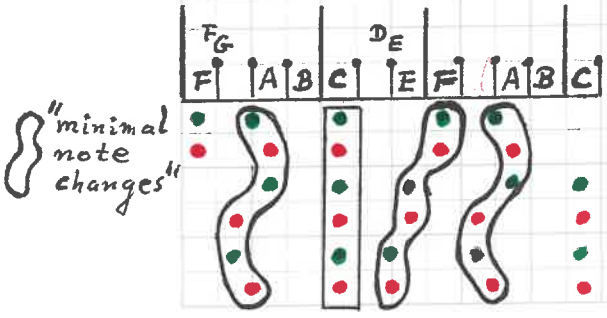
C = C, E, G
G = G, B, D
D = D, F♯, A
A = A, C♯, E
E = E, G♯, B
B = B, D♯, F♯
F♯ = F♯, A♯, C♯
F = F, A, C
B♭ = A♯, D, F
E♭ = D♯, G, A♯
A♭ = G♯, C, D♯
D♭ = C♯, F, G♯

c = C, D♯, G
g = G, A♯, D
d = D, F, A
a = A, C, E
e = E, G♯, B
b = B, D♯, F♯
f♯ = F♯, A♯, C♯
f = F, G♯, C
b♭ = A♯, C♯, F
e♭ = D♯, F♯, A♯
a♭ = G♯, B, D♯
d♭ = C♯, E, G♯

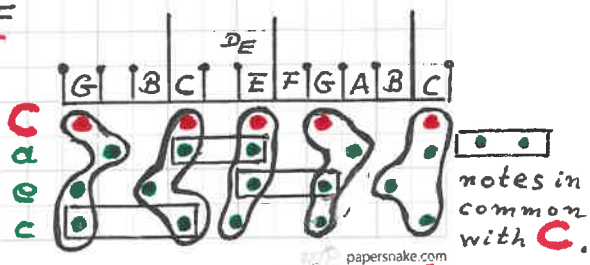
For example, C = C, E, G = {C, E, G} ⇒ CEG, EGC, GCE.



• The left figure shows the TRIAD NEIGHBORHOOD OF THE NOTE C (C, F, A♭, c, f, a) and the TRIAD NEIGBORHOOD OF THE TRIAD C (a, e, c) via the "dual of the Tonnetz." The illustrations below show that these neighborhoods make possible "SMOOTH CHANGES/PROGRESSIONS" OF TRIADS (and keys) - ideal for modulation.



Neighborhood of note C.



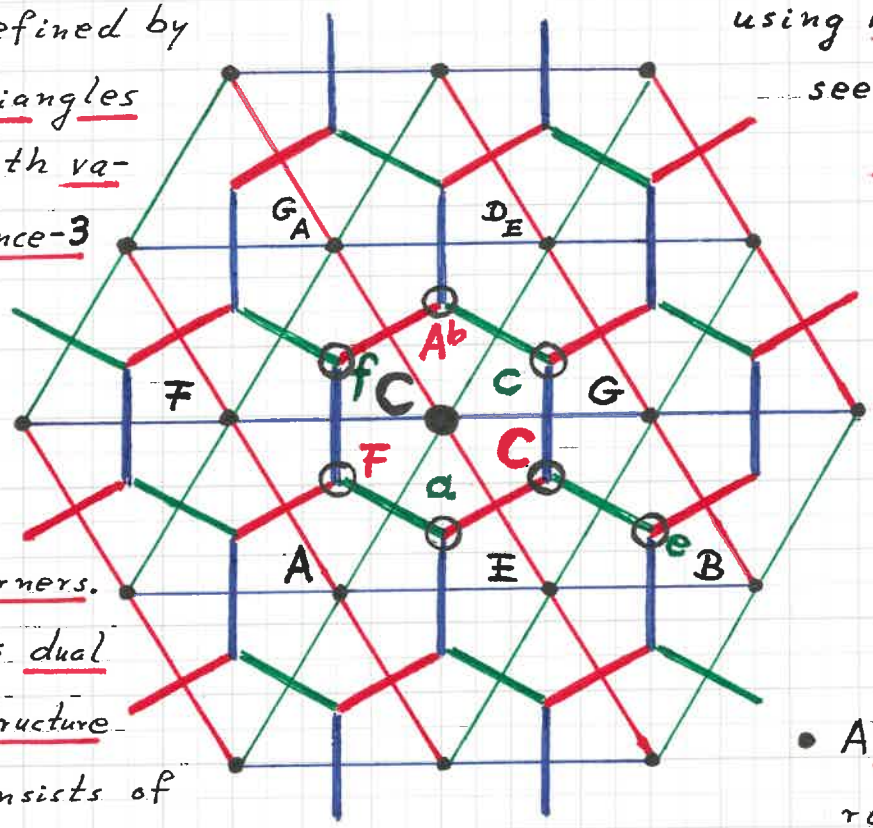
Neighborhood of triad C.

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• Chords in the Tonnetz

11/5/2025

• The Tonnetz is defined by triangles with valence-3 corners. Its dual structure consists of hexagons with valence-6 corners.



• Considering the chromatic scale using halftones, one sees that these structures capture a multitude of relevant intervals:



C→G:	5th
G→C:	4th
C→E:	M3rd
E→C:	m6th
C→D_E:	m3rd
D_E→C:	6th

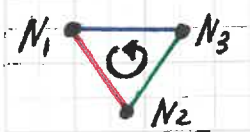
• A Tonnetz point '•' represents a note, having 6 "neighbor notes" — defining 3 major and 3 minor triads sharing the note '•'.

• A point 'O' of the Tonnetz' dual structure represents a triad, having 3 "neighbor triads" — where a major triad has 3 minor "neighbor triads" and a minor triad has 3 major "neighbor triads." A PAIR OF "NEIGHBOR TRIADS" HAS A "TOTAL NOTE DISTANCE" OF EITHER 1 OR 2 HALFTONES.

• Various interesting facts concerning the dual structures are:



The clockwise-oriented triangle with notes N_1, N_2 and N_3 defines the minor triad $N_1 N_2 N_3$ and its two inversions $N_2 N_3 N_1$ and $N_3 N_1 N_2$.



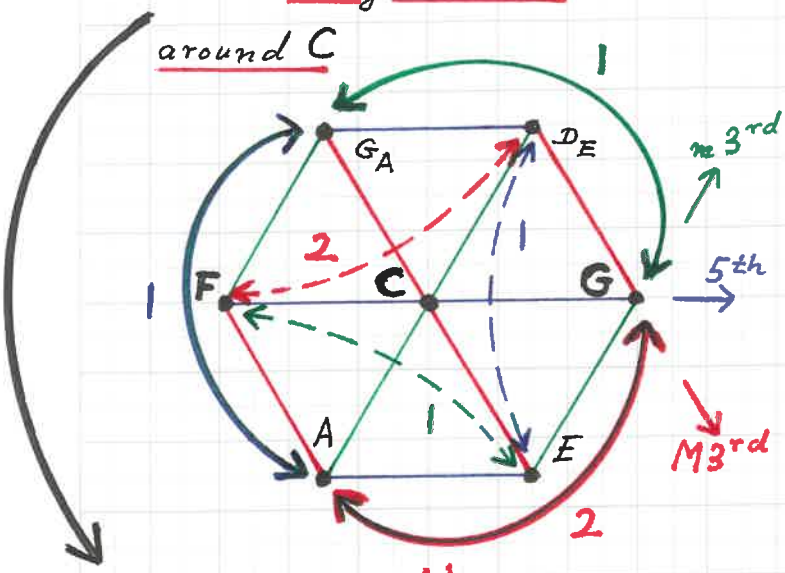
The counter-clockwise-oriented triangle with notes N_1, N_2 and N_3 defines the major triad $N_1 N_2 N_3$ and its two inversions $N_2 N_3 N_1$ and $N_3 N_1 N_2$.

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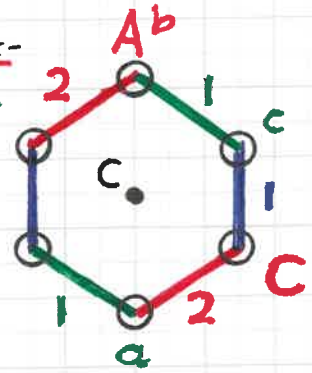
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• Chords in the Tonnetz

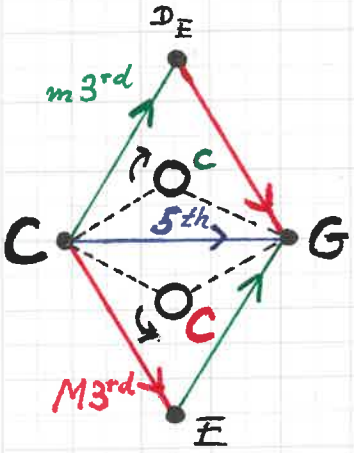
Local neighborhood around C



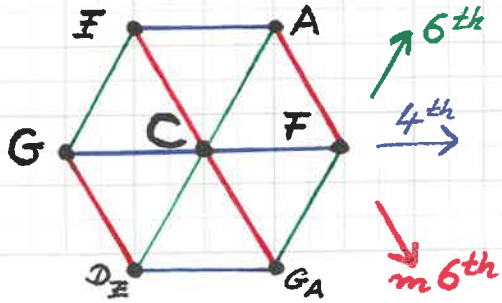
Dual structure of local neighborhood: a hexagon



The left figure shows that the corner points/notes of the hexagon around C have "note distances" of either 1 or 2 halftones — when considering "pairs of every other point/note." Since the set of six triangles/triads around C can be ordered, one can effectively construct/define a "loop of triads around C" (left figure). As already discussed above, the "total note distances" between consecutive triads are 1 or 2 halftones.



The left figure shows the neighborhood of the C major and C minor triads (C and c). The "3-direction network" is defined by the circles of m 3rds, M 3rds and 5ths. The "parallelogram" CCGc (edges shown by broken lines) has the 5th CG as its diagonal — and the minor triad c on its "left side" and the major triad C on its "right side."



The left visualization is obtained by "inverting/reversing directions." One obtains the circles of 4ths (distance = 5 halftones), 6ths (distance = 9 halftones) and m 6ths (distance = 8 halftones).

• Chords in Keyboard Neighborhoods

1/16/2025

The table lists triads $2 \times 4 \times 3 = 24$ chromatic-lexicographically.



Triad			Name	
C	D	F♯ G♭		
		G	C sus 2	
		G♯ A♭		
	D♭ E	F♯ G♭	C dim	
		G	C min	
		G♯ A♭	A♭ dim 1st	
	E	F♯ G♭		
		G	C maj	
		G♯ A♭	C aug	
	F	F♯ G♭		
		G	C sus 4	
		G♯ A♭	F min 1st	
C♯ D♭	D	F♯ G♭		
		G		
		G♯ A♭		
	D♭ E	F♯ G♭		
		G		
		G♯ A♭	D♭ sus 2	
	E	F♯ G♭		
		G	D♭ dim	
		G♯ A♭	D♭ min	
	F	F♯ G♭		
		G		
		G♯ A♭	D♭ maj	

Triads in 9-key/-note neighborhood.

In order to play a "multitude" of triads (in root, 1st inversion or 2nd inversion position) one must also know how to play them "densely and compactly" in just one octave, for example. The left table is based on three "note sets," $\{C, C\sharp/D\flat\}$, $\{D, D\sharp/E\flat, E, F\}$ and $\{F\sharp/G\flat, G, G\sharp/A\flat\}$, and it lists all possible triads — when using one note from each set. When playing the resulting 24 triads on the keyboard, using only keys between the C-key and $G\sharp/A\flat$-key, one can obtain triads played in root or first/second inversion (1st, 2nd). Not all triads are harmonically meaningful or have "standard names."

• Chords in Keyboard Neighborhoods

1/6/2025

Key	Notes	Inversion	C C_D D D_E E F F_G G G_A A A_B B
C	C E G		o o o
a	A C E	1	o o o
G	G B D	2	o o o
e	E G B		o o o
D	D F_G A		o • o
b	B D F_G	1	o • o
A	A C_D E	1	• o o
f#	F_G A C_D	2	• • o
E	E G_A B		o • o
d^b	C_D E G_A		• o •
B	B D_E F_G	1	• • o
a^b	G_A B D_E	2	• • o
F#	F_G A_B C_D	2	• • •
e^b	D_E F_G A_B		• • •
D^b	C_D F G_A		• o •
b^b	A_B C_D F	1	• o •
A^b	G_A C D_E	1	o • •
f	F G_A C	2	o o •
E^b	D_E G A_B		• o •
c	C D_E G		o • o
B^b	A_B D F	1	o o •
g	G A_B D	2	o o •
F	F A C	2	o o o
d	D F A		o o o

24 triads in circle-of-5ths order.
'1' = first, '2' = second inversion.

→ Every note appears in 6 triads.

→ Every note-pair appears in 2 triads.

→ Every triad shares 2 of its notes with 3 other triads.

→ 10 triads are in root position, 7 are in 1st and 7 in 2nd inversion.

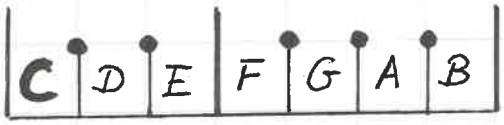
→ The 7 notes defining a key can be "covered" by 3 triads with only

The left table lists the 12 major and 12 minor triads, using the order of the circle-of-fifths. The right column provides a visualization of all 24 triads when played in one octave, using root position, first inversion or second inversion of a triad — to "make it fit" into one octave (on the keyboard). The 24 key patterns shown in the right column make it possible to use an extremely "dense and compact playing style" for these triads — staying within one octave and thus minimizing hand movement across the keyboard. Of course, depending on the **desired** or necessary harmonization, such a compact way to play triads might not always support the intended effects.

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• Chords in Keyboard Neighborhoods 1/7/2025

"Harmonization and Chord/Key Progressions Using Key-inherent Triads or 'Triad Parts'"



Harmonization can be defined and described in many ways, using more or less formal approaches. In the following, basic principles, ideas and "rules" are informally summarized – for the key of C major:

1) The 7 key-inherent triads are I = CEG, ii = DFA, iii = EGB, IV = FAC, V = GBD, vi = ACE, vii° = BDF.
2) Each of these triads must be understood as a root position that also represents its first and second inversions, i.e., I = {CEG, EGC, GCE} etc.
3) Several triad subsets of 3 (or more) triads of the set {I, ii, iii, IV, V, vi, vii°} include (together) all 7 notes of the C major scale.
4) One can establish various triad subsets of 3 (or more) triads for harmonizing the notes of C major – as long as these 3 (or more) triads jointly include all 7 scale notes.
5) Different selections of such subsets of 3 (or more) triads lead to different "colors," "emotions," "effects" when harmonizing a melody (the same melody) with them.
6) The selected triads in such a subset might lead to a PROGRESSION that uses them in some arbitrary sequence or in a repetitive/periodic way.
7) Each of the 7 key-inherent triads can/must also be understood as chords defining "sub-chords," e.g., I = {CEG, EGC, GCE; CE, EC, CG, GC; (EG); C, (E), (G)}.

• Chords in Keyboard Neighborhoods 1/7/2025

Harmonization Principles ...

8) One can view a triad progression as the result of harmonizing a given melody; alternatively, one can understand a melody as the result produced by a given triad progression.

Based on these principles, we consider examples of possible triad progressions for the C major scale:

→ I, IV, V — CEG, FAC, GBD
CG, FAC, GBD
CE, FA, GBD

→ I, vi, V, IV — CEG, ACE, GBD, FAC
CG, AE, GBD, FA
GCE, ACE, GBD, ACF

→ I, ii, V — CEG, DFA, GBD
GCE, FAD, GBD
GC, ADF, BDG

→ vii°, I, IV — BDF, CEG, FAC
BDF, CEG, CFA

→ I, IV, vi, I, V — CEG, CFA, CEA, CEG, BDG

One uses such progressions most commonly in a periodic fashion, e.g., I, IV, V, I, IV, V, I, IV, V, ...

From a combinatorics-driven viewpoint, one is interested in the numbers of possible C major-scale harmonizing triad progressions — consisting of 3, 4, 5, 6 or all maximally possible 7 key-inherent triads. One must keep in mind that one must determine both the set of 3 or more triads and the order defining the specific sequence/cycle of the triads.

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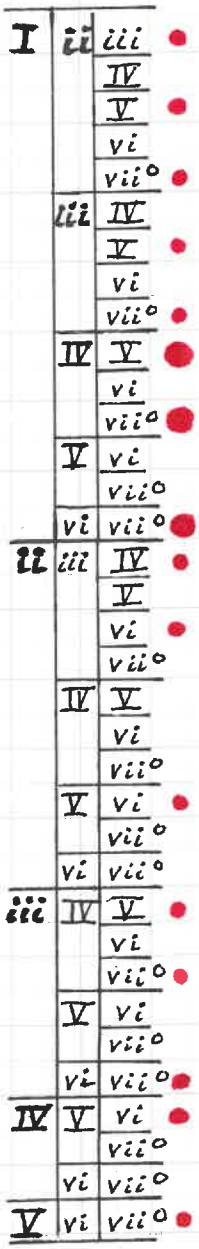
• Chords in Keyboard Neighborhoods 1/8/2025

Harmonization Principles...

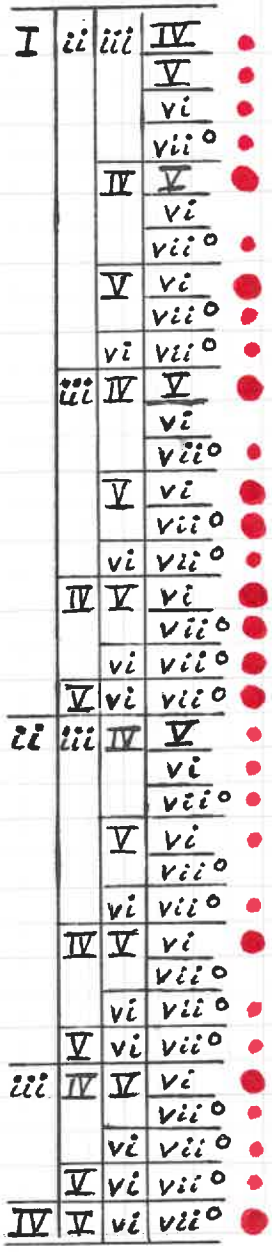
Note. It is often desirable to play the triads (or note subsets) of a progression "densely and compactly" on the keyboard – minimizing finger movements / key distances and also leading to a more "pleasing sound."

We now determine possible triad selections and possible sequences / cycles to play them. The number of possibilities to select 3, 4, 5, 6 or 7 triads from

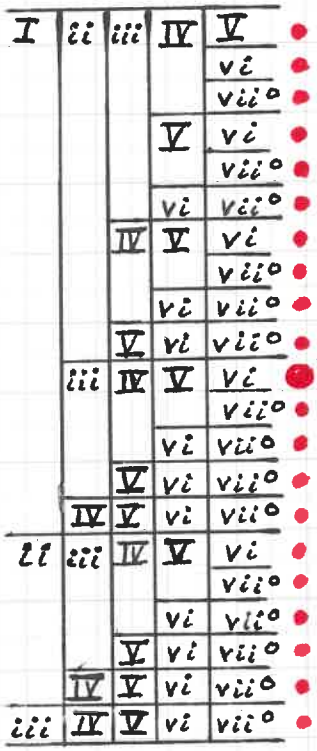
• 3 triads



• 4 triads

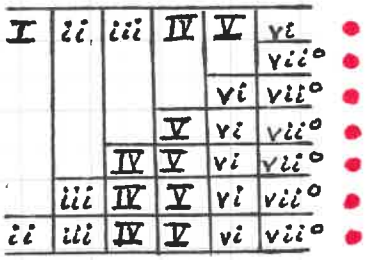


• 5 triads



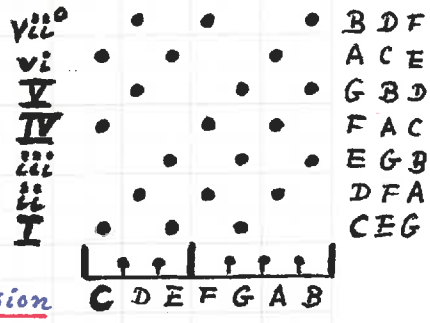
P

• 6 triads



• 7 triads





P Pachelbel's progression

the 7 key-inherent triads is $\binom{7}{3}=35$, $\binom{7}{4}=35$, $\binom{7}{5}=21$, $\binom{7}{6}=7$ and $\binom{7}{7}=1$, respectively. These numbers are the lengths of the five tables on this page. SELECTED TRIADS THAT TOGETHER "COVER" ALL 7 NOTES OF C MAJ. ARE MARKED!

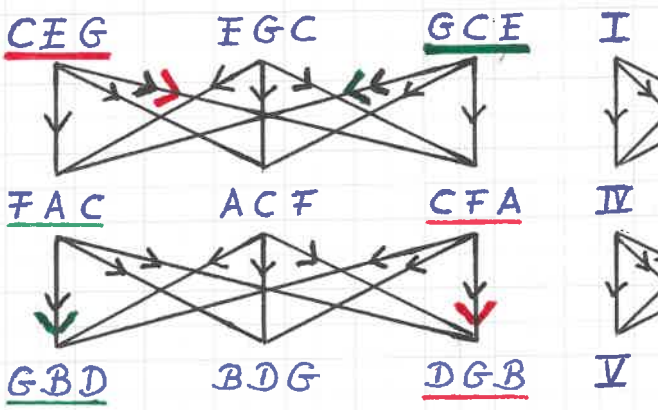
• Chords in Keyboard Neighborhoods 11/9/2025

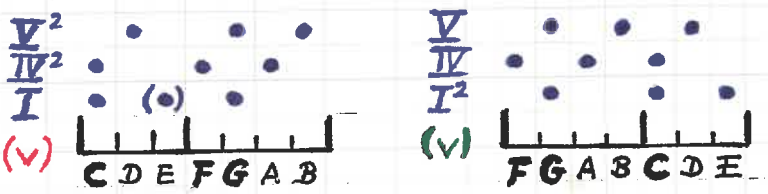
Harmonization Principles...

Notes. 1) The total number of selections of 3, 4, 5, 6 or 7 triads from the 7 key-inherent triads is $35+35+21+7+1=$ **99**. It is important to note that only **16** (of 35) triad selections cover all notes of C major when selecting 3 triads and only **31** (of 35) triad selections cover all notes of C major when selecting 4 triads.

2) A set of k triads can be ordered — and played — in $k!=1\cdot 2\cdot\ldots\cdot k$ ways, e.g., the triad set {I, IV, V} can be ordered in $3!=6$ ways: I-IV-V, I-V-IV, IV-I-V, IV-V-I, V-I-IV, V-IV-I. This fact implies, for example, that the 16 triad selections (when selecting 3 triads) can all be played in 6 ways — defining a total of 96 possibilities to play these 16 triads.

3) One should use a triad's root position and inversions to determine ways to play a triad-progression "compactly and densely" on the keyboard to minimize hand and finger movements. The $3\times 3\times 3=$ **27** possibilities to play the I-IV-V progression with second inversions I^2, IV^2 and V^2 are shown in the figure.





Two possibilities to play I-IV-V densely. I=CEG could be played partially as CG.

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Harmonization Principles...

4) The possibility to "play a triad partially" is powerful. For example, I = CEG can be played as CEG, CG or simply C. Depending on the "harmonic context" of a specific composition, the "human ear/brain can complete the intended triad when played only in part."

5) Many commonly used progressions use a specific triad/chord multiple times. This possibility further increases the number of available progressions to harmonize in a given key. For example, I-IV-V could be "extended" via repeated use of I: I-IV-I-V, I-IV-V-I or I-IV-I-V-I are all viable. The triad progression of "Pachelbel's Canon" also uses triads multiple times. It is defined as **I-V-vi-iii-IV-I-IV-V**.

6) Additional comments concerning progressions/harmonization:

- A very common progression is I-V-vi-IV.
- 12-bar rock and blues progressions usually consist of three "groups," e.g., I-I-I-I; IV-IV-I-I; V-IV-I-I.
- Chord progressions can also include 7th chords — or more complex chords — e.g., V could also be V^7.
- One can harmonize a major scale with 3 major triads, i.e., I (tonic), IV (subdominant) and V (dominant).
- One can analogously harmonize a minor scale with 3 minor triads, i.e., i, iv and v. (Note: 3rd of v often raised by a halftone, to include the "Leitton".* Example: i=ACE, iv=DFA, v=EGB ⇒ V= $EG^{\#}B$, including "Leitton" $G^{\#}$.)

*Leitton = leading tone

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Harmonization Principles...

- A progression chord can **include** a note(s) not belonging to the specific diatonic scale. Often, including such a note(s) "prepares" modulation to another key.
- Typically, progressions have a length of only a few bars.
- Two-triad/-chord progressions are "most simple," and alternating between I and V(7) is commonly done.
- Three-triad progressions can capture all 7 notes of a melody in a specific key. One commonly uses them as four-triad "groups" — to match a "binary rhythm." Examples: I-IV-V-V, I-IV-I-V, I-IV-V-IV.
- ONE CAN USE A MELODY TO DETERMINE A PROGRESSION; ONE CAN USE A PROGRESSION TO DETERMINE A MELODY.
- One can use "groups" of four triads as "building blocks" and append/concatenate them, e.g., (I-IV-V-V)-(I-IV-V-I). This can be done repeatedly, to satisfy the need of a 4-/8-/16-/32-bar melody.
- Blues often uses a "12-bar form" with 3 "groups." A typical progression is (I-I-I-I)-(IV-IV-I-I)-(V-IV-I-I). (Chuck Berry and Little Richard have used it.)
- So-called "1950s progressions" include minor triads/chords. Examples are: I-vi-IV-V and I-vi-ii-V.
- CIRCLE-OF-FIFTHS PROGRESSIONS use "circle adjacency" to define the sequence of triads/chords of a progression. Thus, the Circle makes possible ascending or descending using 5^{th}s/4^{th}s. Examples: vi-ii-V-I = $_{C_{maj}}$ ACE-DFA-GBD-CEG; I-IV-vii°-iii-vi-ii-V-I (full-circle progression).

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Harmonization Principles...

- Sometimes, major triads/chords are used in a progression, instead of the "correct" minor triads/chords, e.g., using I-VI-II-V instead of I-vi-ii-V, when considering the key/scale of C major, for example. Again, this is done to "prepare" modulations.
- Perceptually, harmonization via progressions that "move up/move down a scale step-by-step" is pleasing. ("Pachelbel's Canon" is an example that employs such a principle.)
- All these principles apply equally to major and minor keys. (An example of a descending minor progression is the "Andalusian Cadence," defined as $i-VII-VI-V^{(7)}$ = $\underset{A\,min.}{=}$ ACE - GBD - FAC - EG#B (D). Here, V = EG#B is used instead of v = EGB, since G# is the leading tone in A minor.)

---

7)

	C	D	E	F	G	A	B	
I	•	o		•	o		o		i
ii		•	o		•	o		•	o	ii°
iii		o		•	o		•	o		III
IV	•		o		•	o		•		iv
V		•		o		•	o		•	v
vi	•	o		•		o		•	o	VI
vii°		•	o		•		o		•	VII

"Harmonization table."

The table shown here is a "7×7 matrix" that summarizes compactly the essence of harmonization with triads of the C major scale ('•', '•') and the A minor scale ('o', 'o'). The first row of the table shows that I can be used to harmonize the notes C, E and G ('•') and that i can be used to harmonize the notes A, C and E ('o'). The first column shows that C can be harmonized with the triads I, IV and vi of the C major scale and that C can be harmonized with the triads i, III and VI of the A minor scale—for example.

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Harmonization Principles...

The following examples demonstrate the use of the "harmonization table" on the previous page:

→ Given the progression I-IV-V-I, one can select notes from the "note set sequence" {C, E, G} - {F, A, C} - {G, B, D} - {C, E, G} to define a melody. Of course, when making note selections from one of these "note sets, one can select one, two or all notes in the set - possibly also selecting a note(s) multiple times (defining a simple "one-note-sequence" of a melody or a more complex "triad-/chord-sequence").

→ Given a sequence of single notes defining a melody, e.g., C-E-F-B-D-A, one can define implied triad progressions — using either the C major scale or the A minor scale for harmonization. Considering the six-note sequence example, the two sequences of "triad sets" are:

C major harmony: {I, IV, vi} - {I, iii, vi} - {ii, IV, vii°} - {iii, V, vii°} - {ii, V, vii°} - {ii IV vi}.

A minor harmony: {i, III, VI} - {i, III, v} - {ii°, iv, VI} - {ii°, v, VII} - {ii, iv, VII} - {i, iv, VI}.

THE PROBLEM OF DETERMINING A "GOOD" — OR "THE BEST" — TRIAD PROGRESSION TO HARMONIZE A MELODY (WHEN ONLY A NOTE SEQUENCE IS GIVEN) IS DIFFICULT! FOR EXAMPLE, GIVEN THE NOTE SEQUENCE / MELODY $N_1-N_2-N_3-N_4-N_5-N_6-N_7-N_8$, HOW DOES ONE DECIDE THAT $N_1-N_2 \Rightarrow$ I; $N_3-N_4-N_5-N_6 \Rightarrow$ V; $N_7-N_8 \Rightarrow$ I IS "GOOD"? IS $N_1-N_2 \Rightarrow$ III; $N_3-N_4-N_5-N_6 \Rightarrow$ VII; $N_7-N_8 \Rightarrow$ III ALSO "GOOD"???

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8) A "simple harmonization algorithm." The problem of determining a (formally) viable harmonization / progression for a given melody is discussed in more detail in the following.

- It is assumed that a melody is given — a sequence of notes (some of them possibly played in las/with chords).
- The key of the melody is known. (C major / A minor is assumed and used in the example below.)
- We "move a window" of fixed width over the melody-notes from "left" to "right," e.g., we move a window that contains 2 or 3 or 4 or... consecutive notes and "slide it" over the melody notes.
- For each window position "along the line of melody notes," we determine for how many of the k considered notes lying inside the window, one could use triad I, ii, iii, IV, V, vi, vii° or i, ii°, III, iv, v, VI or VII to harmonize the k notes. ⇒ "Triad counters" for each position!
- Since the key of the melody is known, one might eventually be interested in a harmonization that only uses a simple set of triads / chords to define a / the possibly underlying progression — e.g., I-IV-V-I or i-iv-v-i.

(• Using such a "sliding window" method resembles a type of "discrete convolution.")

- For each window position along the melody, determine the triad / triads with maximal "triad-counter" values.
- We can now determine sequences ("lines" / "poly-lines") of triads "along the melody" connecting those with maximal "triad-counter" values. (Consider all 7 or only certain triads.)

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- The triad sequence(s) resulting from this "purely formal" algorithmic method for determining harmonizing triads (and a progression(s)) might not be "perceptually pleasing" (to the ear). Thus, one also must consider "requirements of rhythm and human perception" to determine, modify and finalize a harmonization — that often must support a desired effect as well.
- We use the first 14 notes (of the melody) of "Alle Vögel sind schon da" von Heinrich Hoffmann von Fallersleben to demonstrate the described technique. (Originally, the song's key is D major; we have transposed it to C major to establish the 14-note opening sequence.)

- Used window width for counter value computation: 4 notes

C maj	A min	C	E	G	C	A	C	A	G	F	G	E	C	D	C
I	i	1x 1x	2x 2x	3x 2	4x 3	3x 3x	3x 3x	2 4x	2 3x	2 2	2x 1	3x 1	3x 2	3x 2	3x 3x
ii	ii°					1	1	2	2	2 1	2 1	1 1	1 1	1 1x	1 1
iii	III	1x	1 2x	2 3x	2 4x	2 3x	1 3x	2	1 2	1 1	2x 2x	3x 3x	2 3x	2 3x	1 3x
IV	iv	1x	1	1	2	2 0	3x 1	2x 2	3x 2	3x 2	2 2	1 1	1 1	1 1	2 1
V	v		1	1 2	1 2	1 2	1 1		1 1	1 1	2x 2x	2 3x	1 2	1 2	1 1
vi	VI	1x 1x	2x 1	2 1	3 2	3x 2	3x 3	3x 2x	3x 3x	2 3x	1 2	1 1	1 2	2 1	3x 2
vii°	VII			1	1	1	1		1	1 1	1 2x	1 2	1 1	2 1	1 1

- "Correct harmonization":

C E G C → I
A C A → IV
G → I
F G → V
E C → I
D → V
C → I

The counter value **0** is not included in the table; all "blank spaces" in the table represent this value. Under each note, maximal counter values of triads of the C maj (A min) key are marked with an (x); The associated triads can be viewed as "harmonization candidates."

⇒ ONE MUST USE THIS TABLE TO DERIVE VIABLE CHOICES FOR HARMONIZATION — POSSIBLY MODIFYING WINDOW WIDTH.

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• We use the same 14-note example to demonstrate that it is HELPFUL / IMPORTANT TO APPLY THIS METHOD TO THE GIVEN NOTES BY TRAVERSING THEM FORWARD – FROM LEFT TO RIGHT – AND BACKWARD – FROM RIGHT TO LEFT to make the resulting "combined" "triad-counter" values DIRECTION-INDEPENDENT. In the following example we use a note-window width of 2 when moving forward or backward.

C maj	A min	C	E	G	C	A	C	A	G	F	G	E	C	D	C
I	i	1 1 2 2	2 2 2 1	2 1 2 1	2 1 1 2	1 2 1 2	1 2 1 2	1 2 1 1	1 1 1	1 1	1 2 1	2 1 2 2	2 2 1 1	1 1 1 1	1 1 1 1
ii	ii°				1	1 1	1 1	1 1	1 1 1	1 1 1 1	1 1		1 1	1 1 1 1	1 1
iii	III	1 1 2	1 2 2 2	2 2 1 2	1 2 1	1 1	1 1	1 1 1	1 1 1 1	1 1 1 1	1 1 2 2	2 2 1 2	1 2 1	1 1	1 1
IV	iv	1 1	1	1	1 2 1	2 1 2 1	2 1 2 1	2 1 1 1	1 1 1 1	1 1 1 1	1 1	1	1 1 1	1 1 1 1	1 1 1
V	v	1	1 1 2	1 2 1 1	1 1			1 1	1 1 1 1	1 1 1 1	1 1 1 2	1 2 1	1 1	1 1	1
vi	VI	1 1 2 1	2 1 1	1 1 1	1 1 2 2	2 2 2 2	2 2 2 2	2 2 1 1	1 1 1	1 1 1	1 1	1 2 1	2 1 1 1	1 1 1 1	1 1 1 1
vii°	VII		1	1 1	1				1 1 1	1 1 1 1	1 1 1 1	1		1 1 1 1	1 1

black number-pairs (top): window from left to right

blue number-pairs (bottom): window from right to left

• adding top and bottom rows of numbers

• maximal values of sums underlined in each column (blue)

C maj	A min	C	E	G	C	A	C	A	G	F	G	E	C	D	C
I	i	3 3	4 3	4 2	3 3	2 4	2 4	2 3	2 1	2	3 1	4 3	3 3	2 2	2 2
ii	ii°				1	2	2	2	2 1	2 2	1 1		1 1	2 2	1 1
iii	III	1 3	3 4	3 4	1 3	2	2	1 2	2 2	2 2	3 3	3 4	1 3	2	2
IV	iv	2	1	1	3 1	4 2	4 2	3 2	2 2	2 2	1 1	1	2 1	2 2	2 1
V	v	1	1 3	2 3	1 1			1 1	2 2	2 2	2 3	1 3	1 1	2	1
vi	VI	3 2	3 1	2 1	3 3	4 4	4 4	3 3	1 2	2	1 1	3 1	3 2	2 2	2 2
vii°	VII		1	2	1			1	1 2	2 2	1 2	1	1 1	2 2	1 1

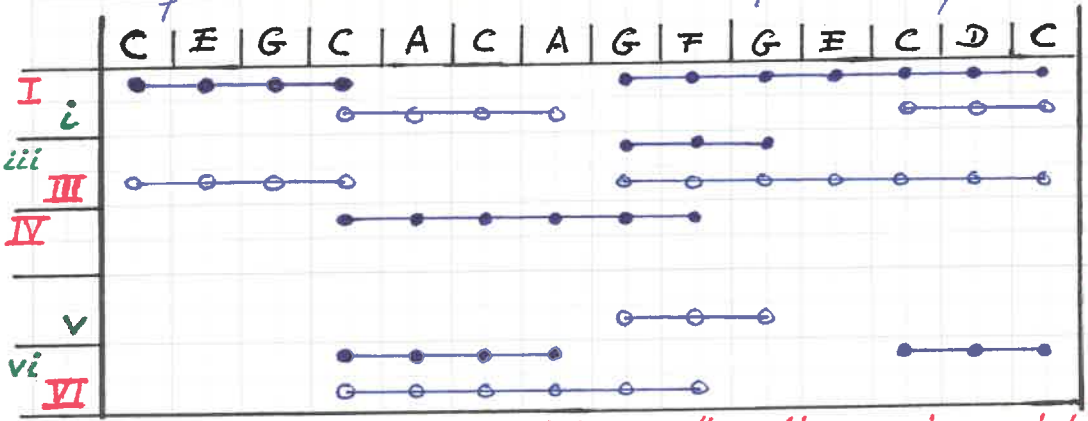
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C maj	A min	C	E	G	C	A	C	A	G	F	G	E	C	D	C
I	i	●○	●	●	●○	○	○	○	●	●	●	●	●○	●○	●○
ii	ii°								●	●○				●○	
iii	III	○	○	○	○				●○	●○	●○	○	○	○	○
IV	iv				●	●	●	●	●○	●○				●○	●
V	v								●○	●○	○			●	○
vi	VI	●			●○	●○	●○	●○	○	○			●	●○	●
vii°	VII								○	●○				●○	

This table marks the "winning and formally viable" triads that are the key-inherent triads of C major ('●') and A minor ('○') that are the "candidate triads" for harmonizing the given 14-note sequence. Considering the alternative to harmonize with C major-inherent triads, the number of possible triad sequences is therefore $2\times1\times1\times3\times2\times2\times2\times5\times6\times2\times1\times2\times6\times3 = 2^8\cdot3^4\cdot5 = 256\cdot81\cdot5$.

(The number of possibilities is the same when considering A minor-inherent triads.) Thus, an effective, efficient and practical method for determining a "truly acceptable and pleasing harmonizing triad sequence is necessary. Obviously, the $256\cdot81\cdot5 = 103680$ "triad candidate" sequences cannot be "tested practically." For example one can identify "spans of consecutive notes" that can be harmonized by one of the "candidate triads" marked '●'/'○'



This table shows such "spans" – those where at least three consecutive notes share a common "candidate triad."

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The "spans" shown in the figure at the bottom of the previous page suggest the following harmonization possibilities:

C maj. triads: CEGC → I, CACAGF → IV (or CACA → vi), GFGECDC → I

A min. triads: CEGC → III, CACAGF → VI (or CACA → i), GFGECDC → III

The method generates the "corresponding harmonization sequences" for C major and A minor triads, i.e., I-IV(or vi)-I and III-VI (or i)-III.

We also provide the results produced by the described method for a "window of width 1." In this extreme setting, one only needs to determine whether a note is or is not part of a particular triad. Again, in order to determine a formally viable harmonization of the 14 notes with a small (or minimal) number of "spans," one must first identify those triads that can be used to harmonize largest-possible subsets of consecutive notes, and then combine/connect/concatenate (some or all of) these triads such that all 14 notes are harmonized.

C A	C	E	G	C	A	C	A	G	F	G	E	C	D	C
I i	• o	• o	•	• o	o	• o	o	•		•	• o	• o		• o
ii ii°					•		•		• o				• o	
iii III	o	• o	• o	o	o		o	• o		• o	• o	o		o
IV iv	•			•	• o	•	• o		• o			•	• o	•
V v		o	• o					• o		• o	• o		•	
vi VI	• o	•		• o	• o	• o	• o		• o		•	• o		• o
vii° VII			o					o	•	o			• o	

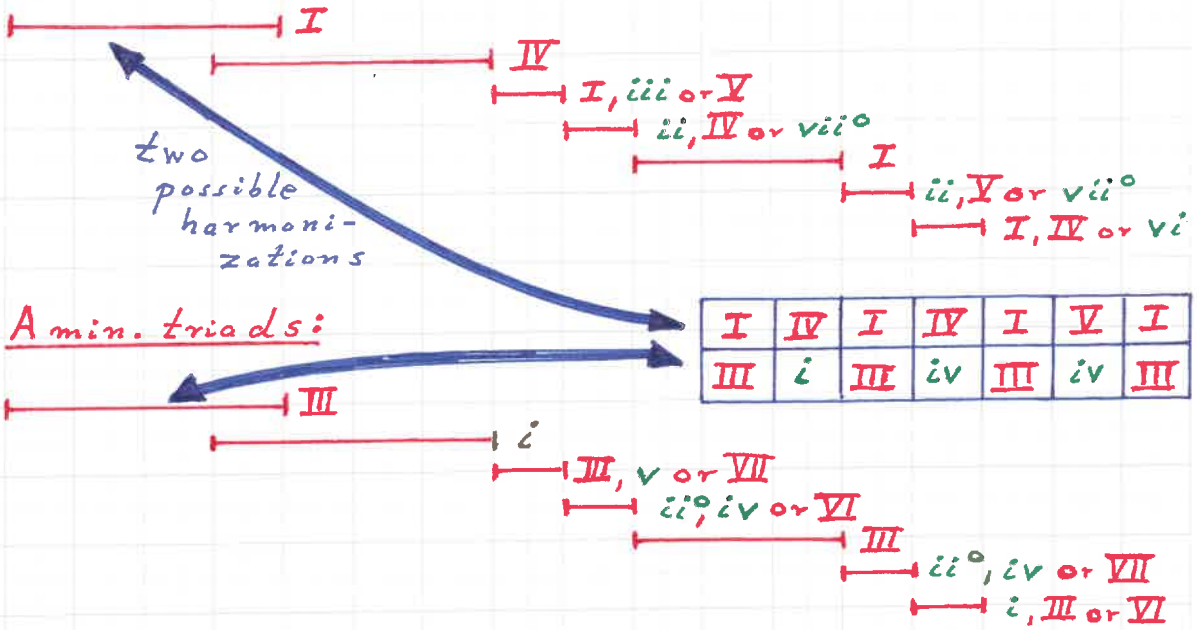
In the left table, a mark '•' or 'o' is made when a triad can be used to harmonize the note in the corresponding column.

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"Spans" can be identified by carefully looking at the table at the bottom of the previous page. These "spans" lead to the following formally viable harmonizations:

C maj. triads:

C E G C A C A G F G E C D C



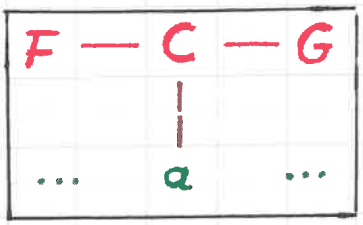
Note. First, the "1-note spans" can all be harmonized with three possible triads. Second, some "spans" overlap, i.e., the 4th note, C, can be harmonized with **I** or **IV** (or III or i). Thus, "span overlaps" provide various harmonization possibilities. Third, the "correct harmonization" (as published) is CEGC→I, ACA→IV, G→I, FG→V, EC→I, D→V, C→I; the harmonization that is closest to this sequence that can be constructed from the C major triad "spans" shown above is CEGC→I, ACA→IV, G→I, F→IV, GEC→I, D→V, C→I. Thus, the only difference in harmonization is FG→V and EC→I versus F→IV and GEC→I. Overall, one can see that the described method can generate "relatively small harmonization sequences" [illegible] select a practically desirable one(s).

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Additional comments concerning this harmonization method:

→ Considering harmonization with key-inherent triads of C major (or any other major scale), and knowing that the key of a composition/melody is C major, one could (initially) use a harmonization maximizing the use of C major triads I = CEG, IV = FAC and V = GBD — since I is the tonic, IV is the subdominant and V is the dominant (triad). (Analogously, one should use this approach also when harmonizing a melody in a known minor key — prioritizing triads i, iv and v.)

→ Considering harmonization with C major triads again, the circle of fifts and its "neighborhoods" also suggest harmonization triads that are "close":



C = I = CEG (tonic) has F = IV = FAC (subdominant) and G = V = GBD (dominant) as its "closest major neighbors"; but Amin = a = vi = ACE defines the "closest minor neighbor" of C and should thus also be used. Thus, {I, IV, V, vi} also defines a good 4-triad harmonization with triads inherent to the C major scale.

→ In order to incorporate desirable "tension" in a harmonization, one can also consider the use of 7th chords. For example, one has the possibility to "extend" I, IV and V, e.g., using $I^7 = CEGB^b$, $IV^7 = FACE^b$ and $V^7 = GBDF$, and $vi^7 = ACEG$.

→ Ultimately, rhythm and desired "harmonic effects" will impact the final harmonization greatly!

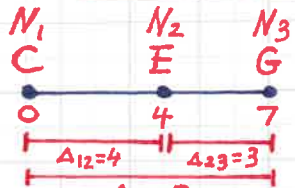
→ One must further decide whether it is "better" to harmonize with a small or large number of triad changes for a melody.

• Chord Consonance and Dissonance

1/26/2025

While it is difficult — or even impossible — to define an "objective quantitative measure for qualitative terms like consonance and dissonance", it is still important to establish at least a "musically acceptable ranking" of 2-note chords, triads and possibly 4-note chords to order chords from "most beautiful" to "most ugly."

One can employ a "minimalistic" form of 2-note chord ranking by considering semi-tone distance from a base/root note and assign "consonance values" between **0** (least consonant, "most ugly") to **12** (most consonant, "most beautiful"):



0	1	2	3	4	5	6	7	8	9	10	11	12
C	C_D	D	D_E	E	F	F_G	G	G_A	A	A_B	B	C
12	1	4	6	7	9	0	10	5	8	2	3	11

Semi-tone indices of octave (top); semi-tones from C to C (middle); "consonance values" for 13 2-note chords on C.

Δ23	CO												
12	34												
11	18	15											
10	16	6	17										
9	28	11	15	25									
8	22	14	11	14	23								
7	32	16	22	18	20	30							
6	12	11	9	14	9	12	11						
5	30	10	23	20	24	20	12	30					
4	26	17	11	23	19	24	9	20	23				
3	24	14	19	12	23	20	14	18	14	25			
2	20	11	15	19	11	23	9	22	11	15	17		
1	14	6	11	14	17	10	11	16	14	11	6	15	
0	36	14	20	24	26	30	12	32	22	28	16	18	34
CO: Triad consonance	0	1	2	3	4	5	6	7	8	9	10	11	12

Symmetry of triad consonance CO: CO (Δ12, Δ23) ≡ CO (Δ23, Δ12)

… path from 6 to 30

Δ12

• THE GOAL IS THE ESTABLISHMENT OF A VIABLE, SIMPLE AND "MUSICALLY ACCEPTABLE RELATIVE RANKING OF CHORDS" — NOT OF AN "ABSOLUTE, CHORD-SPECIFIC NUMERICAL VALUE OF CONSONANCE."

(1) The above table provides a "generally accepted ranking."

(2) The left table provides the sum of the three "consonance values" of the three 2-note "sub-chords" of triads $N_1N_2N_3$: N_1N_2, N_2N_3 and N_1N_3.

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• Chord Consonance and Dissonance

It is interesting to list the highest-ranking triads (with root note C) resulting from the simple ranking method described on the previous page:

	CO	Δ_{12}	Δ_{23}	triad	
1	30	5	7	CFC	Subdominant
		7	5	CGC	Dominant
2	25	3	9	C^{D}EC	m3rd on C
		9	3	CAC	m3rd on A
3	24	4	5	CEA	A min. 1st inv.
		5	4	CFA	F maj. 2nd inv.
4	23	2	5	CDG	C sus2
		5	2	CFG	C sus4
		3	4	C^{D}EG	C min. root pos.
		4	3	CEG	C maj. root pos.
		4	8	CEC	M3rd on C
		8	4	C^{G}AC	M3rd on G$_A$
⋮					

Ranking does NOT consider chords for which $\Delta_{12}=0$ or $\Delta_{23}=0$ (chords with "double" or "triple" notes).

• Note. The left table lists the triads with "consonance values" 30, 25, 24 and 23 — using ranks 1, 2, 3 and 4, respectively. This rank order might not be in agreement with other existing and accepted rank orders. In order to produce other "meaningful" rank orders, one can consider the use of weights applied to the 13 2-note chord "consonance values" or, alternatively, the definition of 13 2-note chord "consonance values" where the value sequence is not the uniform, equidistant sequence 0, 1, 2, ..., 12.

• Additional Notes. Chord inversions, 4-note chords, progressions.

(1) Triad/Chord inversions have different "consonance values" — and it should be this way. Depending on one harmonical and musical view of root position and inversions, one must derise one's definition and computation of "consonance values" accordingly.

(2) One can generalize the described method for triads $N_1 N_2 N_3$, "$N_1 \leq N_2 \leq N_3$," to 4-note chords $N_1 N_2 N_3 N_4$, "$N_1 \leq N_2 \leq N_3 \leq N_4$," where one uses interval lengths $\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{13}, \Delta_{24}, \Delta_{14}$.

• Chord Consonance and Dissonance 1/27/2025

(3) Often, one prefers a "gradual and smooth progression" when moving from a (possibly "very dissonant") chord to another (possibly "very consonant") chord — e.g., when moving from a "tension chord" to a "resolution chord." One can use the table / function $CO(\Delta_{12}, \Delta_{23})$ to determine a smooth path / progression from a start triad to a termination triad, by making only minimal changes to "consonance values" when moving from triad to triad. We consider two examples.

	Δ_{12}	Δ_{23}	CO	triad
start	1	1	6	$C\,{}^{C}_{D}\,D$
	1	2	11	$C\,{}^{C}_{D}\,{}^{D}_{E}$
	1	3	14	$C\,{}^{C}_{D}\,E$
	1	4	17	$C\,{}^{C}_{D}\,F$
	2	3	19	$C\,D\,F$
	3	2	19	$C\,{}^{D}_{E}\,F$
	4	3	23	$C\,E\,G$
	5	4	24	$C\,F\,A$
end	7	5	30	$C\,G\,C$

	Δ_{12}	Δ_{23}	CO	triad
start	1	1	6	$C\,{}^{C}_{D}\,D$
	2	2	15	$C\,D\,E$
	4	3	23	$C\,E\,G$
	5	4	24	$C\,F\,A$
end	7	5	30	$C\,G\,C$

The left tables define two paths from the START triad $C\,{}^{C}_{D}\,D$ to the END triad $C\,G\,C$. The START triad is highly dissonant while the END triad is highly consonant. The first progression involves nine triads; the second progression uses only five triads by making "larger jumps" when moving from triad to triad. Both progressions define "non-decreasing paths," i.e., "consonance values" never decrease when moving from a triad to the next. Such a "monotonicity rule" seem reasonable to enforce. Two relevant questions arise:

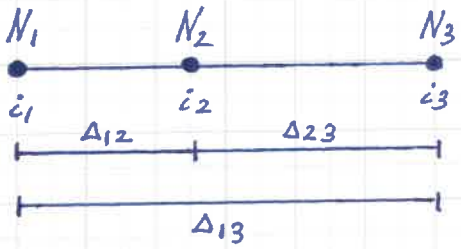
→ Should a "good progression" consist of many or few triads?

→ Are such purely "consonance value"-based progressions musically "nice and acceptable"?

• Chord Consonance and Dissonance

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• **Note.** One can view the described method for calculating "consonance values" as a method for defining consonance (and also dissonance) via multivariate functions. In the case of triads, one uses a bivariate function; in the case of 4-note chords, one uses a trivariate function.

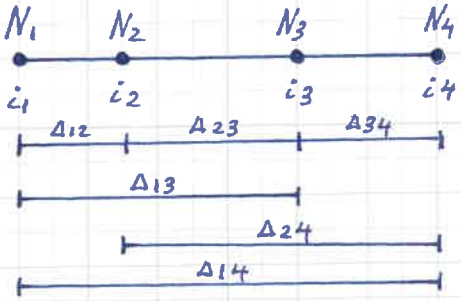


Triad and its parameters.

Triads. A triad is understood as $N_1 N_2 N_3$ with associated (semi-tone) indices i_1, i_2 and i_3 — where, for example, piano keys could be indexed from 1 to 88.

It is assumed that $i_1 \leq i_2$ and $i_2 \leq i_3$. Thus, the relevant 2-note (diad) chord "interval lengths" are $\Delta_{12} = i_2 - i_1$, $\Delta_{23} = i_3 - i_2$ and $\Delta_{13} = i_3 - i_1 = \Delta_{12} + \Delta_{23}$. The described function for computing "consonance values" is a bivariate function:

$$CO(N_1 N_2 N_3) = CO(i_1, i_2, i_3) = CO(\Delta_{12}, \Delta_{23}, \Delta_{13}) =$$
$$= CO(\Delta_{12}) + CO(\Delta_{23}) + CO(\Delta_{12} + \Delta_{23}) = CO(\Delta_{12}, \Delta_{23}).$$



4-note chord and parameters.

4-note chords. The generalization of the function defined for triads yields a trivariate function for chords $N_1 N_2 N_3 N_4$, where $i_1 \leq i_2$, $i_2 \leq i_3$ and $i_3 \leq i_4$ and $\Delta_{12} = i_2 - i_1$, $\Delta_{23} = i_3 - i_2$, $\Delta_{34} = i_4 - i_3$, $\Delta_{13} = i_3 - i_1 = \Delta_{12} + \Delta_{23}$, $\Delta_{24} = i_4 - i_2 = \Delta_{23} + \Delta_{34}$ and $\Delta_{14} = i_4 - i_1 = \Delta_{12} + \Delta_{23} + \Delta_{34}$. Thus, the function becomes:

$$CO(N_1 N_2 N_3 N_4) = CO(i_1, i_2, i_3, i_4) = CO(\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{13}, \Delta_{24}, \Delta_{14}) =$$
$$= CO(\Delta_{12}) + CO(\Delta_{23}) + CO(\Delta_{34}) + CO(\Delta_{12} + \Delta_{23}) +$$
$$+ CO(\Delta_{23} + \Delta_{34}) + CO(\Delta_{12} + \Delta_{23} + \Delta_{34}) = CO(\Delta_{12}, \Delta_{23}, \Delta_{34}).$$

⇒ As a consequence, one can **plot/visualize** "consonance values" for a 2D or 3D domain (plane or space).

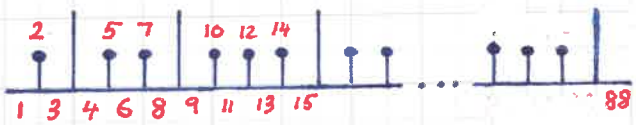
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• Chord Consonance and Dissonance

1/28/2025

• In order to make effective and musically meaningful use of the proposed method for computing "consonance values," one must address and solve a crucial problem:

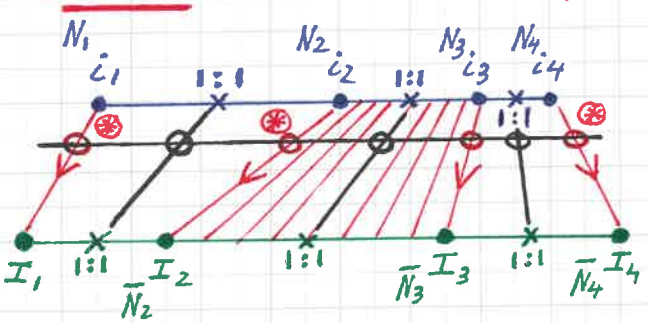
➔ **How** can one establish chord progressions that **BOTH** satisfy and use traditional ways of smooth, gradual harmonic progression principles of movement **AND** consider and incorporate "consonance values" of chords to define "best-possible progressions"?



The left figure is merely provided for context: A "good method" for computing "consonance values" for an 88-key keyboard must consider and use the actual "physical key indices" 1...88 since (i) the same chords when played in different octaves of the keyboard should/must have different "consonance values" and (ii) a 2-note chord N_1 N_2 should/must have different "consonance values" depending on the octaves where the note N_1 and the note N_2 are played. (Example: N_1 = C, N_2 = D, i_1 = 4, i_2 = 6 should/must have a different "consonance value" than N_1 = C, N_2 = D, i_1 = 4, i_2 = 18.)

• **Note.** One can also define and – if helpful – utilize a PIECEWISE LINEAR MAPPING of the indices of semi-tones.

Piecewise Linear mapping I of a 4-note chord



⊛ "Smooth" movement of note N_1,... to note $\bar{N}_1$,...

The left figure illustrates the map

$$I = \frac{i_{k+1} - i}{i_{k+1} - i_k} I_k + \frac{i - i_k}{i_{k+1} - i_k} I_{k+1}, \quad i_{k+1} \neq i_k, \; k = 1 \ldots 3.$$

N_k is mapped to $\bar{N}_k$, and semi-tone index i ($i_k < i < i_{k+1}$) is mapped to semi-tone index I (after rounding) defines a note between $\bar{N}_k$ and $\bar{N}_{k+1}$

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• Chord Progressions and Circles

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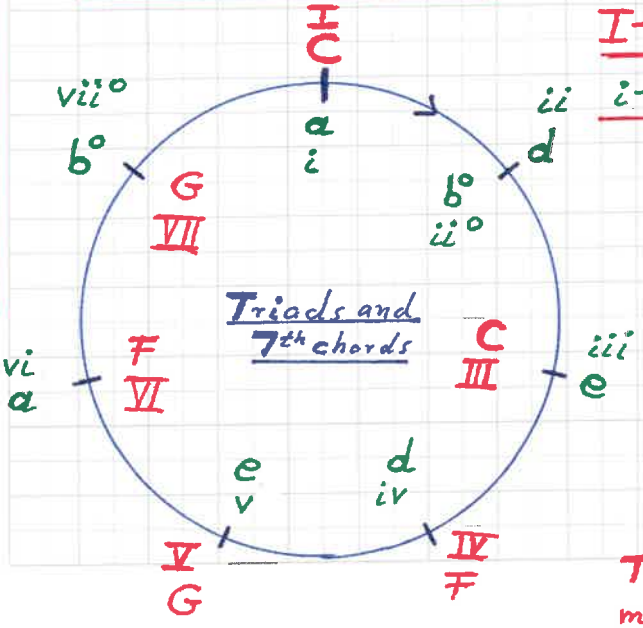
It is extremely important to realize that chord progressions can be established by employing various kinds of CIRCLES, defined by different ways to view periodic / cyclic characteristics. Here, we focus on **triads** and **7th chords**. One must keep in mind that an established circular order (forward and backward) applies to chords in their root positions and to all their inversions. Further, a circular order defines the progressions of both major and (three) minor keys/scales. Key-/scale-inherent chords only use the notes of the respective underlying key/scale; circle-of-fifths-defined chords, on the other hand, use all 12 notes of the chromatic scale. In the following, we summarize these three circles: (1) the "circle of 2nds" using key-/scale-inherent chords only; (2) the "harmonic circle of 5ths" using key-/scale-inherent chords only; and (3) the "standard circle of 5ths" using chords with notes from the entire 12-note chromatic scale.

Ex.:

C	E	G
D	F	A
E	G	B
F	A	C
G	B	D
A	C	E
B	D	F
C	E	G

(1) Circle of 2nds.

This circle defines the progressions I-ii-iii-IV-V-vi-vii°-I and i-ii°-III-iv-v-VI-VII-i. The keys of C maj. and A min. are shown as example keys. For these two keys, the chords are:



$I = C = \{$ CEG-EGC-GCE
CEGB-EGBC-GBCE-BCEG ; ...

$i = a = \{$ ACE-CEA-EAC
ACEG-CEGA-EGAC-GACE ; ...

Thus, the circle representation implies maj. and minor key cycles and all inversions of chords

• Chord Progressions and Circles 1/30/2025

Again, the seven triads / 7th chords defined by the "circle of 2nds" uses only the 7 notes of a given key / scale (allowing to raise notes by a semi-tone to also capture harmonic and melodic minor scales). We provide the chords for steps vii° and VII as well:

b° = { BDF - DFB - FBD / BDFA - DFAB - FABD - ABDF } = vii°

G = { GBD - BDG - DGB / GBDF - BDFG - DFGB - FGBD } = VII

Ex.:

C E G
G B D
D F A
A C E
E G B
B D F
F A C
C E G

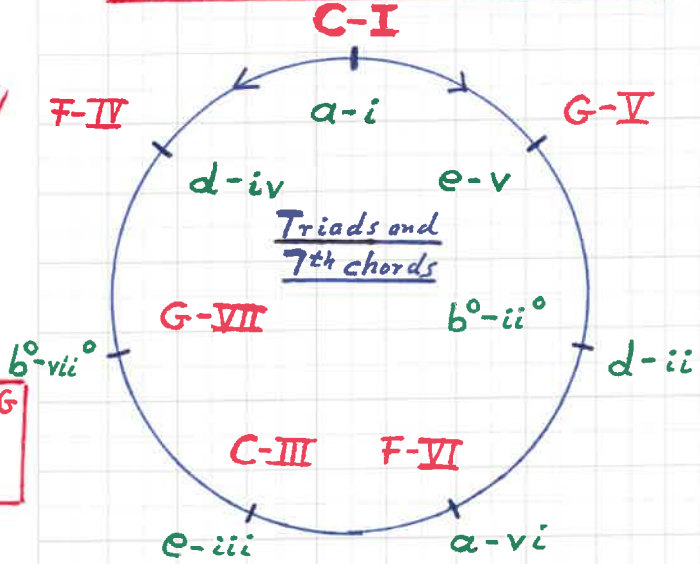
C E G
F A C
B D F

Ex.:

C
G
D
A
E
B
F#
C#
G#
D#
A#
E#
B#=C

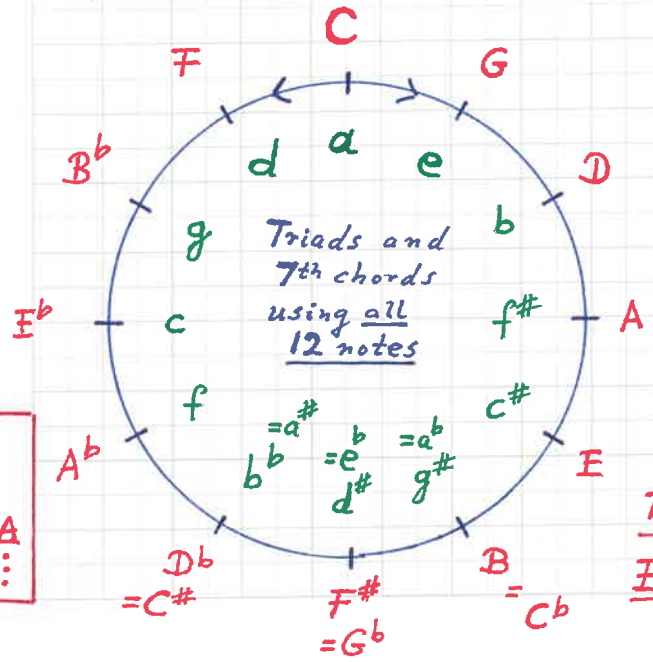
C E G
G B D
D F# A
⋮

(2) Harmonic circle of 5th s.



This circle defines the (forward) progressions I-V-ii-vi-iii-vii°-IV-I and i-v-ii°-VI-III-VII-iv-i. Again, these progressions are shown for the keys / scales of Cmaj. and Amin. The triads and 7th chords (and their inversions) associated with C, F, G, d, e, a and b° only use the seven notes of the Cmaj. and Amin. scales. Thus, the difference between circle (1) and circle (2) is movement by 2nds vs. 5ths.

(3) Standard circle of 5th s.



In contrast to circles (1) and (2), this circle always moves forward / backward by seven semi-tones — until the "key used as start key" is reached again. Here, C is at the top / center (= start and end key). Triads / 7th chords use the notes of all 24 keys

Example triads: E ⇒ EG#B - G#BE - BEG#; c# ⇒ C#EG# - EG#C# - G#C#E.

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• Chord Progressions and Circles

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The circle of fifths uses fifths to establish an order of the 12 major (12 minor) keys as "neighbor keys" only differ by one semi-tone (and thus one key on the keyboard) only. Thus, when modulating from a key to one of its two neighbors on the circle of fifths, the progression from a key (and its associated key-inherent chords) to a "neighbor key" (and its associated key-inherent chords) is smooth. We consider the "key neighborhood" of F-C-G (d-a-e) as an example.

keys: ^{b}s ⟷ keys: $^{\#}$s

F, d			C, a			G, e		
V	CEG	VII	I	CEG	III	IV	CEG	VI
vi	DFA	i	ii	DFA	iv	V	DF$^{\#}$A	VII
vii$^{\circ}$	EGB^{b}	ii$^{\circ}$	iii	EGB	v	vi	EGB	i
I	FAC	III	IV	FAC	VI	vii$^{\circ}$	F$^{\#}$AC	ii$^{\circ}$
ii	GB^{b}D	iv	V	GBD	VII	I	GBD	III
iii	ACE	v	vi	ACE	i	ii	ACE	iv
IV	B^{b}DF	VI	vii$^{\circ}$	BDF	ii$^{\circ}$	iii	BDF$^{\#}$	v

Key-inherent triads

The left table lists the seven key-inherent triads of C, F and G and of the parallel/natural minor keys a, d and e, respectively. The "neighbor key" triads have the semi-tone differences B versus B^{b} and F versus F$^{\#}$. Thus, "neighbor keys" exhibit maximal triad similarity — and key modulations F⟷C and C⟷G imply minimal triad changes. (The same is true, of course, for the parallel/natural minor keys.)

• From C to G with "pivot" GBD:

CEG (=C I)
↓
GBD (=C IV)
↓
DF$^{\#}$A (=G V)
↓
GBD (=G I)

• Underlined triads are "pivot triads" – for smooth modulation.

In the following, we consider simple modulations.

vii$^{\circ}$ BDE (0) ⟷ BDF$^{\#}$ ⟷ BDG (1) I
(C)
DFB (1) ⟷ DF$^{\#}$B ⟷ DGB (2) (G)
FBD (2) ⟷ F$^{\#}$BD ⟷ GBD (0)

(0) = root position
(1), (2) = first, second inversion

C ⟷ G

The example shown here is a modulation C⟷G, where C triad vii$^{\circ}$ is changed by altering F to F$^{\#}$ (leading tone), moving to G.

• Chord Progressions and Circles

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Other possible modulations are:

vii° E G B^b (0) ↔ E G B ↔ E G C (1) I (C)
(F) G B^b E (1) ↔ G B E ↔ G C E (2)
B^b E G (2) ↔ B E G ↔ C E G (0)

Example shows modulation F↔C, using B^b → B → C.

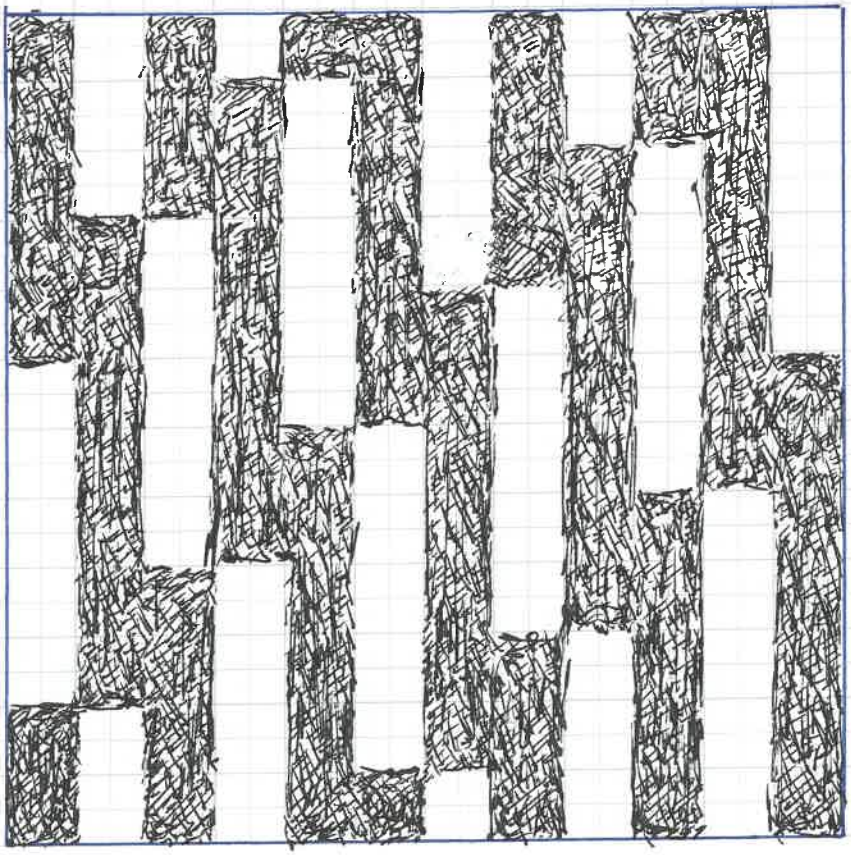
VII G B D (0) ↔ G B $D^\#$ ↔ G B E (1) i (e)
(a) B D G (1) ↔ B $D^\#$ G ↔ B E G (2)
D G B (2) ↔ $D^\#$ G B ↔ E G B (0)

Example shows modulation a↔e, using D → $D^\#$ → E.

VII C E G (0) ↔ C E $G^\#$ ↔ C E A (1) i (a)
(d) E G C (1) ↔ E $G^\#$ C ↔ E A C (2)
G C E (2) ↔ $G^\#$ C E ↔ A C E (0)

Example shows modulation d↔a, using G → $G^\#$ → A.

Of course, the musical, harmonic and "sound" qualities of such "theoretically defined" modulations must be properly evaluated. Further, modulations for minor keys involving minor keys are more complex as one must consider natural, harmonic and melodic minors.



The left illustration shows the pattern obtained when filling in the "squares of physical keyboard keys" played for the 12 (24) keys.

• Chord Progressions and Circles

What are different ways to modulate from C major to G majpr ?

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AI Overview

To modulate from C major to G major, the most common method is to use a "pivot chord," like the A minor chord (which is the vi chord in C major and the ii chord in G major), allowing a smooth transition by sharing common tones between the keys; you can also directly introduce the dominant chord of G major (D major) to clearly establish the new key, or use a sequence of chords that gradually lead to the G major key by incorporating chromatic alterations.

Key approaches to modulating from C major to G major:

Pivot chord method:

- Use the A minor chord (vi in C major, ii in G major) as a pivot chord, progressing from C major chords to A minor, then smoothly transition to G major chords by utilizing the shared tones.

Direct modulation with dominant chord:

- At the end of a phrase in C major, introduce the dominant chord of G major (D major) to clearly signal the key change.

CEG -> GBD -> DF#A -> GBD

Chromatic modulation:

- Use chromatic alterations in chords to gradually move towards the G major key, like adding an F sharp to a C major chord to create a more chromatic sound.

Sequential modulation:

- Transpose a chord progression up a fifth (the interval between C and G) to reach G major.

Important points to consider:

- **Harmonic function:** When choosing pivot chords, consider their harmonic function in both keys to create a logical progression.
- **Smooth transition:** Use smooth voice leading between chords to avoid jarring key changes.
- **Cadence:** Use a strong cadence in the new key (G major) to solidify the modulation.

Generative AI is experimental.

Modulating to Closely Related Keys (Ma
Modulation ...
Nov 15, 2021 — number six that could be use
three and c number seven is no good it's got
YouTube · Music Matters

Modulation Basics - Chord Progressions
In this harmonic outline, the A minor chord is
keys: C major and G major and consequently
Chord Progressions in Tonal Music

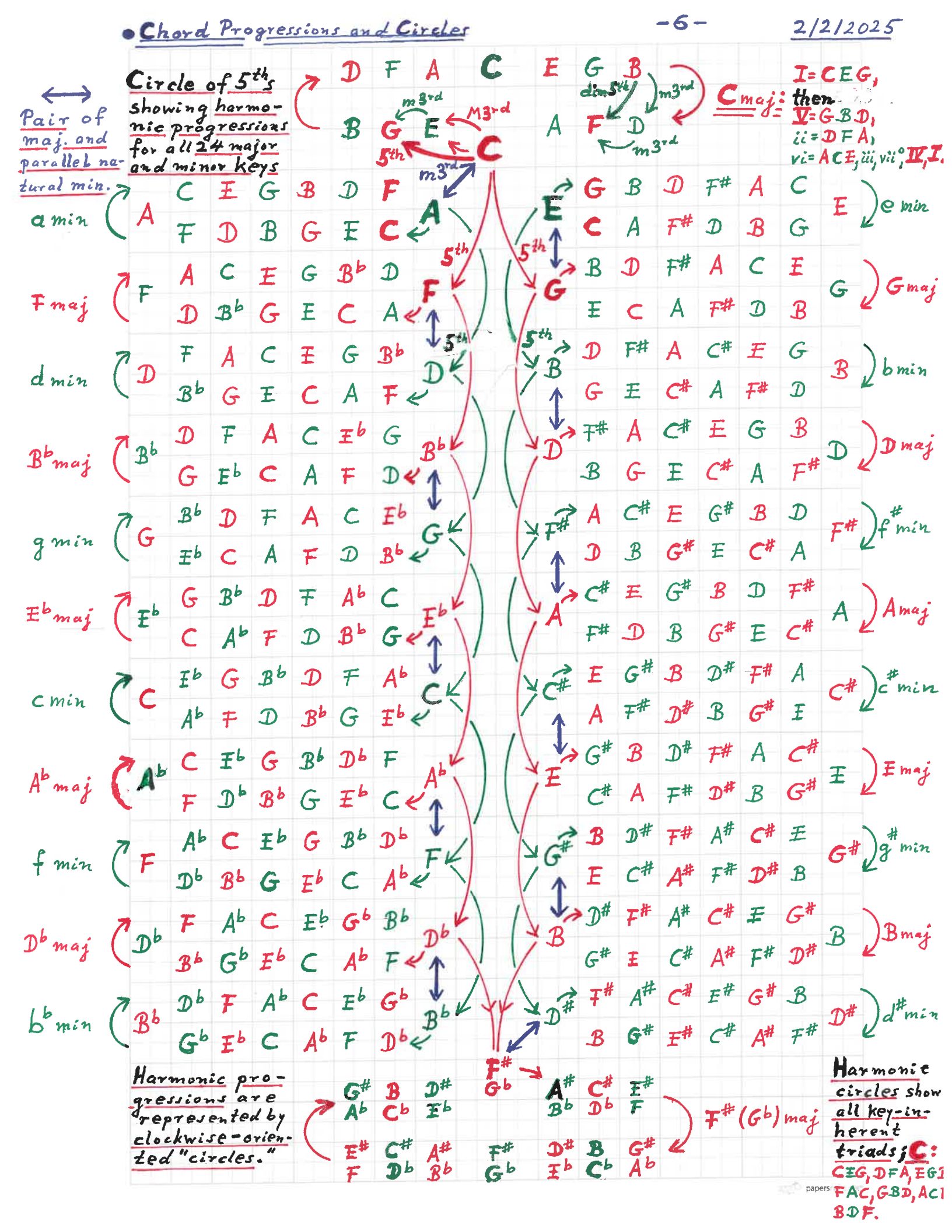
17. Common Chord / Pivot Chord Modul
Pressbooks
These examples only show how to smoothly
move from one key to another. * From a majo
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Chord Progressions and Circles

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• Chord Progressions and Circles 2/3/2025

G major

Key-inherent triads

2nd		5th	
I	GBD	I	GBD
ii	ACE	V	DF#A
iii	BDF#	ii	ACE
IV	CEG	vi	EGB
V	DF#A	iii	BDF#
vi	EGB	vii°	F#AC
vii°	F#AC	IV	CEG
I	GBD	I	GBD

Moving by 2nds

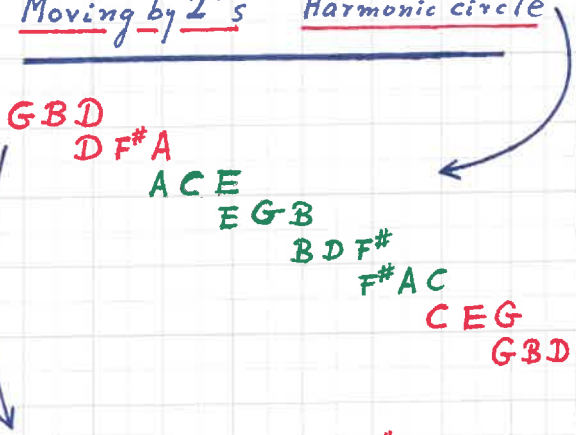
GBD
DF#A
ACE
EGB
BDF#
F#AC
CEG
GBD

GBDF#ACEGBDF#ACEG
I V ii vi iii vii° IV

7th chords defined by circle:

GBDF#
DF#AC
ACEG
EGBD
BDF#A
F#ACE
CEGB

Harmonic circle



e minor

Key-inherent triads

i	EGB	i	EGB
ii°	F#AC	v	BDF#
III	GBD	ii°	F#AC
iv	ACE	VI	CEG
v	BDF#	III	GBD
VI	CEG	VII	DF#A
VII	DF#A	iv	ACE
i	EGB	i	EGB

Moving by 2nds

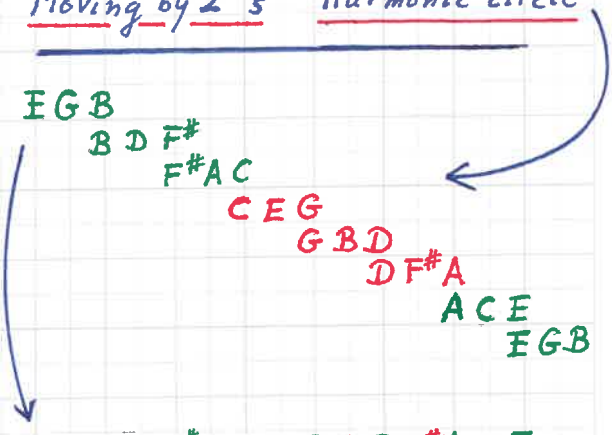
EGB
BDF#
F#AC
CEG
GBD
DF#A
ACE
EGB

EGBDF#ACEGBDF#ACE
i v ii° VI III VII iv

7th chords defined by circle:

EGBD
BDF#A
F#ACE
CEGB
GBDF#
DF#AC
ACEG

Harmonic circle



Representation of circle of fifths – with "inserted minors":

Cb Gb Db Ab Eb Bb F C G D A E B F# C#
eb bb f c g d a e b f# c# g# d# a# e#

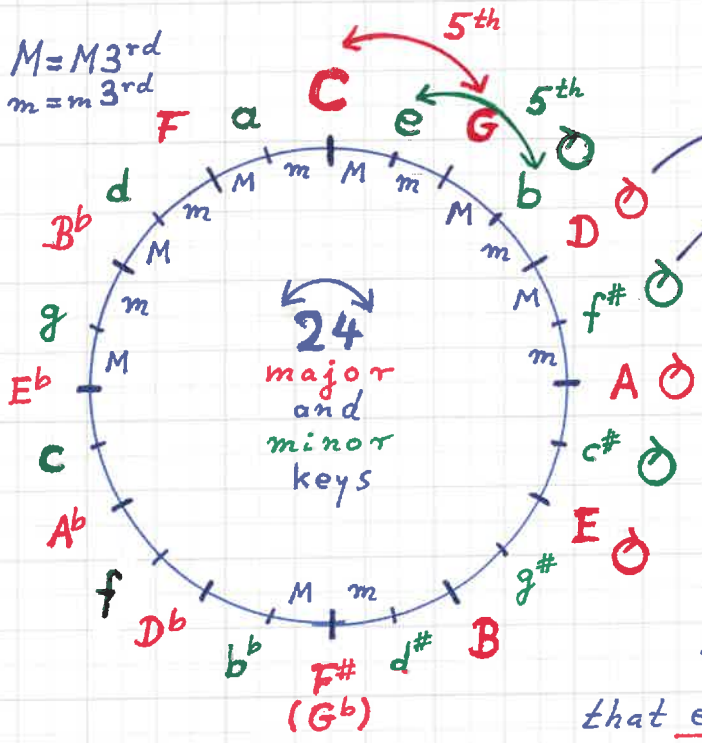
Implied, captured maj. triads: CEG, GBD, DF#A, AC#E, ...

Implied, captured min. triads: ACE, EGB, BDF#, F#AC#, ...

⇒ This representation defines maj. triads by triples  (e.g.,) and min. triads by triples  (e.g.,).

• Chord Progressions and Circles

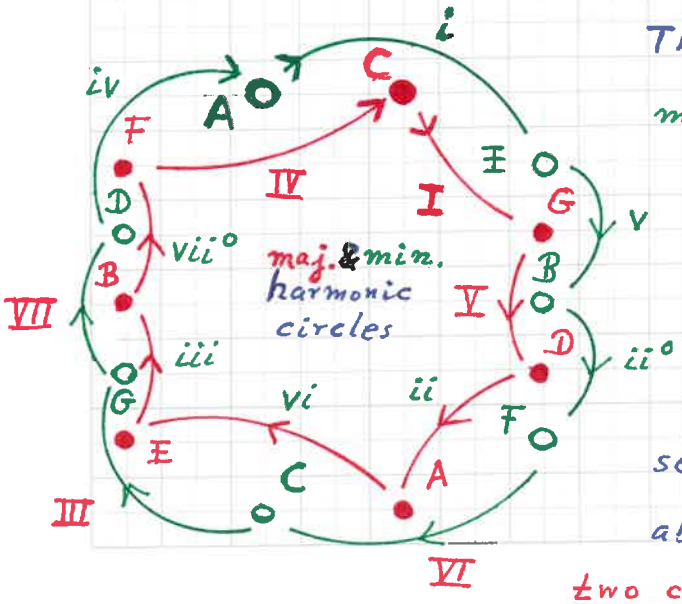
2/3/2025



I-V-ii-vi-iii-vii°-IV-I

i-v-ii°-VI-III-VII-iv-i

The circle of fifths shown here uses a representation that includes all 24 major and minor keys. Further, the left figure indicates that every key has an associated harmonic circle, ↻ or ↺, consisting of the seven key-inherent triads using only the notes of the specific key. This 24-key representation of the circle of fifths — where a major key is placed next to its parallel natural minor key — makes it possible to directly "see" all major and minor triads: CEG, GBD, DF#A, AC#E, EG#B, BD#F#, ...; ACE, EGB, BDF#, F#AC#, C#EG#, G#BD#, ... Also, 7th chords are implied by four consecutive keys/key names.



The left figure shows major and minor harmonic circles, using the specific circles of C major and a minor, to provide context. (These circles are only defined in terms of their Roman numeral sequences next to the figure shown above.) This representation "integrates" two circles and captures M3rds and m3rds.

Modulation via PIVOT chords uses chords that are common chords of the two chords considered for modulation. PIVOT chords serve as "portals" making possible a smooth transition from one chord to another. In the following, we consider the keys of the circle of fifths and indicate all the chords that are shared by two neighbor keys.

• Example: The common triads of C and G are

CI = GIV,
Ciii = Gvi,
CV = GI, $C^b = B$
Cvi = Gii, $G^b = F^\#$
i.e., $C^\# = D^b$
CEG,
EGB,
GBD,
ACE.

key	C	C_D	D	D_E	E	F	F_G	G	G_A	A	A_B	B	C
C	I		ii		iii	IV		V		vi		vii°	I
G	IV		V		vi		vii°	I		ii		iii	IV
D		vii°	I		ii		iii	IV		V		vi	
A		iii	IV		V		vi		vii°	I		ii	
E		vi		vii°	I		ii		iii	IV		V	
B		ii		iii	IV		V		vi		vii°	I	
F♯		V		vi		vii°	I		ii		iii	IV	
D♭	vii°	I		ii		iii	IV		V		vi		vii°
A♭	iii	IV		V		vi		vii°	I		ii		iii
E♭	vi		vii°	I		ii		iii	IV		V		vi
B♭	ii		iii	IV		V		vi		vii°	I		ii
F	V		vi		vii°	I		ii		iii	IV		V
C	I		ii		iii	IV		V		vi		vii°	I

The left table shows the seven key-inherent triads for the 12 major keys. For example, the row for the key of D maj. lists vii°, I, ii, iii, IV, V, vi. This is the triad list C♯EG, DF♯A, EGB, F♯AC♯, GBD, AC♯E, BDF♯. Thus, these seven triads include the two accidentals of D maj., i.e., F♯ and C♯.

• IT IS POSSIBLE TO MODULATE FROM ANY KEY TO ANY OTHER KEY - VIA A SEQUENCE OF KEY PAIRS.

THE FOUR TRIADS THAT ARE SHARED BY TWO "NEIGHBOR KEYS" — LISTED IN TWO "NEIGHBOR ROWS" IN THE TABLE — APPEAR IN COLUMNS WHERE ONE SEES A PAIR OF TWO MAJOR OR OF TWO MINOR TRIADS. THE SYMBOL | INDICATES PAIRS.

• Modulation

The table of PIVOT triads also shows that triads having a "distance of two" in the circle of fifths only have two chords in common. For example, the keys of C maj. and D maj. only share C iii (= G vi) = D ii = EGB and C V (= G I) = D IV = GBD. Triads with a "distance larger than two" no longer share major or minor triads; thus, in order to smoothly modulate from one key to another key having a "large distance," one must perform a sequence of pivot triad modulations — of "distance two" or "distance one."

(It is also important to realize that the three triads C I – G I – D I = CEG – GBD – DF#A $\hat{=}$ $\underline{\underline{CEG}}$ $\underline{\underline{D}}$ F#A permit the transition between C maj. and G maj. and D maj. — via the pivot triad G I = GBD.)

BASIC MODULATION PRINCIPLES

→ Modulating a key to a "neighbor key" in the circle of fifths — or to a key that is "close" — is "simple" as only a small number of #s or bs must be considered: C ↦ G: F ⇝ F#; G ↦ C: F# ⇝ F; C ↦ F: B ⇝ Bb; F ↦ C: Bb ⇝ B.

→ Pivot-chord modulation supports "smooth" transitions. It is based on identifying chords — or possibly parts of chords — that are shared by the two keys considered for modulation. One ends with a cadence of the first key, moves to the shared pivot chord to add (or subtract) an accidental (# or b), and one starts with a cadence of the new, second key.

→ Modulation of minor keys should generally be done with the harmonic-minor scale, since this minor scale has a lead note — unlike the natural, relative minor scale. The harmonic-minor scale makes possible smooth HARMONIC transitions.

→ One can easily visualize the "linked" nature of triads/chords when using the circle-of-fifths order:

C E G D F♯ A E G♯ B F♯ A♯ C♯ G♯ B♯ D♯ A♯ C♯♯ E♯ B♯ D♯♯ F♯♯
G B D A C♯ E B D♯ F♯ C♯ E♯ G♯ D♯ F♯♯ A♯ E♯ G♯♯ B♯

→ Modulation in a key's "neighborhood" generally involves the triads/chords on steps I, IV, V, vi (major keys) to enable smooth pivot-based modulations:

$F_{I,IV,V,vi}$ = FAC, B♭DF, CEG, DFA; $C_{I,IV,V,vi}$ = CEG, FAC, GBD, ACE;
$G_{I,IV,V,vi}$ = GBD, CEG, DF♯A, EGB. ⇒ Possibilities for C↦G or C↦F.

→ Seventh chords and also 9th chords also support smooth modulations. For each of the seven notes of a key, one consider the (up to five) 7th and 9th chords that have a given note as root. For example, for the key of C major, one has these possibilities:

CEGB(D), DFAC(E), EGBD(F), FACE(G), GBDF(A), ACEG(B), BDFA(c);
CEGB♭(D), DFAC♭(E♭), EGBD♭(F), FACE♭(G), GBDF♭(A♭), ACEG♭(B♭), BDFA♭(c);
...CE♭G♭B♭♭ ≙ CE♭G♭A, ..., BDFA♭.

These and other 7th and 9th chords enable chord resolution that make possible smooth transition to a new key.

C maj. and G maj. have triads C, G, e and a in common

→ CEG-DFA-EGB-FAC-GBD-ACE-BDF-CEG ≙ C d-e-F-G-a-b°-C ≙
CEG-DF♯A-EGB-F♯AC-GBD-ACE-BDF♯-CEG C-D-e-f♯°-G-a-b-C

≙ 

is a compact "graph representation" of the 7 triads of C maj. and G maj.

→ One can use an equally compact representation to also show shared-chord behavior of three subsequent (major) keys in the circle of fifths. We consider Cmaj., Gmaj., Dmaj.:

Cmaj.: C E G | D F A | E G B | F A C | G B D | A C E | B D F | C E G

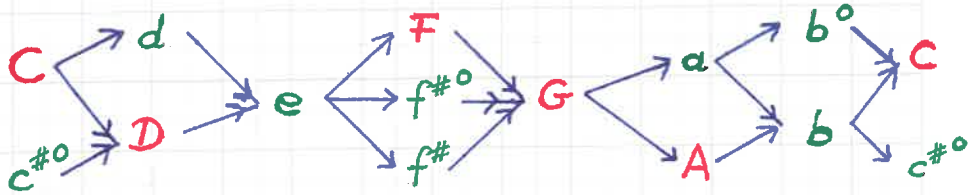
Gmaj.: C E G | D F♯ A | E G B | F♯ A C | G B D | A C E | B D F♯ | C E G

Dmaj.: C♯ E G | D F♯ A | E G B | F♯ A C♯ | G B D | A C♯ E | B D F♯ | C♯ E G

≙ chord/triad names

C	d	e	F	G	a	b°	C
C	D	e	f♯°	G	a	b	C
c♯°	D	e	f♯	G	A	b	c♯°

≙ graph representation with "merged" common triads



≙ Roman numerals associated with all triads

I	ii	iii	IV	V	vi	vii°	I
IV	V	vi	vii°	I	ii	iii	IV
vii°	I	ii	iii	IV	V	vi	vii°

The compact "directed graph representation" of the triads of the three keys visualizes possible transitions/modulations via triads that are are shared by two or three keys. The directed graph edges indicate how one might establish a modulation by using a "triad path" (from "left to right" in the visualized scenario) that passes through a shared "triad node in the graph," moving to another key. As before, the pairs or triplets of triads shared by two or three keys, respectively, are indicated via '|'.

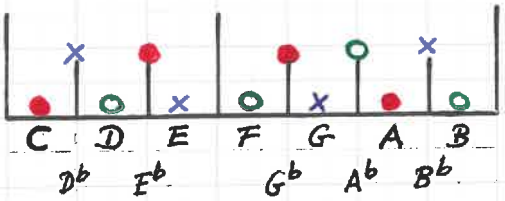
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➔ Modulation via the use of shared/partially shared 7th chords (or even 9th chords) opens up many possibilities, since these "tension chords" make possible the utilization of multiple possible resolutions to various new keys to be transitioned to. Further, one can consider (at least) five such 7th chords for any given root (root ≙ key). Of particular interest and importance in this context is the FULLY DIMINISHED 7th CHORD: This 7th chord (and its three inversions) can be "harmonically interpreted in several ways" and can be resolved to multiple resolution chords/keys. We consider the three fully diminished 7th chords: CE^bG^bA, FA^bBD and GB^bD^bE. (Note. The Cmaj7, Fmaj7 and Gmaj7 chords are CEGB, FACE and GBDF#. Turning these three chords into fully diminished chords requires one to "apply the operator 'b' once to the second and third notes and twice to the fourth note of the chords, leading to $CE^bG^bB^{bb} = CE^bG^bA$, $FA^bC^bE^{bb} = FA^bBD$ and $GB^bD^bF^b = GB^bD^bE$. Each one of these chords is a stack of three minor 3rds. Together, the notes of these three 7th chords cover all 12 notes of a "chromatic octave," and thus the chords do not have any note in common. This fact can be visualized by indicating the chords' notes (•, ○, x) in the left figure. Chord inversions are:

CE^bG^bA, E^bG^bAC, G^bACE^b, ACE^bG^b;

FA^bBD, A^bBDF, $BDFA^b$, DFA^bB;

GB^bD^bE, B^bD^bEG, D^bEGB^b, EGB^bD^b.



3 fully dim. 7th chords cover all 12 notes of an octave.

→ For simplicity, we will write these three 7th chords slightly differently, to avoid the need of using the '#' and 'b' symbols. We use $^{C}_{D}$, $^{D}_{E}$, $^{F}_{G}$, $^{G}_{A}$ and $^{A}_{B}$:

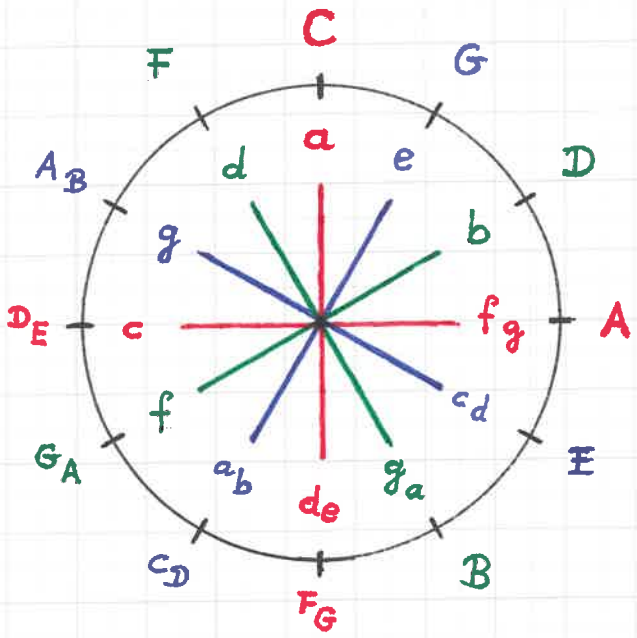
$C E^{b} G^{b} A \hat{=} C^{D}_{E} {}^{F}_{G} A$, $F A^{b} B D \hat{=} F^{G}_{A} B D$, $G B^{b} D^{b} E \hat{=} G^{A}_{B} {}^{C}_{D} E$.

One can assume that each of these three 7th chords can be resolved to possibly eight triads/keys. Resolutions are:

$C^{D}_{E} {}^{F}_{G} A$: C maj, $^{D}_{E}$ maj, $^{F}_{G}$ maj, A maj; c min, $^{d}_{e}$ min, $^{f}_{g}$ min, a min ;

$F^{G}_{A} B D$: F maj, $^{G}_{A}$ maj, B maj, D maj; f min, $^{g}_{a}$ min, b min, d min ;

$G^{A}_{B} {}^{C}_{D} E$: G maj, $^{A}_{B}$ maj, $^{C}_{D}$ maj, E maj; g min, $^{a}_{b}$ min, $^{c}_{d}$ min, e min .



The left figure shows the circle of fifths, where the simplified notation for notes/triads/keys is used again — '#' and 'b' are avoided. The outer letters denote the 12 major keys; the inner letters denote the 12 minor keys.

IT IS RELEVANT TO OBSERVE THAT THE 24 RESOLUTIONS OF THE THREE FULLY DIMINISHED 7th CHORDS PRODUCE A PERFECTLY SYMMETRIC PATTERN IN THE CIRCLE-OF-FIFTHS REPRESENTATION! Again, these 7th-chord resolutions serve modulation to a new key.

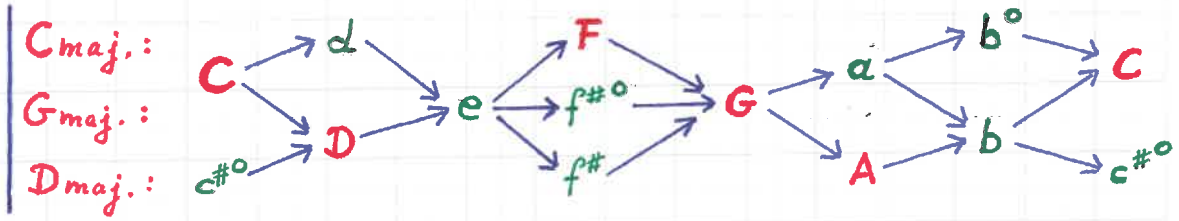
→ Since the fully diminished 7th chord and its three inversions are all defined by m 3rds, one can use/view any of its four notes as its root/its key. For example, the "note sequence" ... $C - {}^{D}_{E} - {}^{F}_{G} - A - C - {}^{D}_{E} - {}^{F}_{G} - A - C - {}^{D}_{E} - {}^{F}_{G} - A - C$... stacks m 3rd-generating the chords $C^{D}_{E} {}^{F}_{G} A$, $^{D}_{E} {}^{F}_{G} A C$, $^{F}_{G} A C^{D}_{E}$, $A C^{D}_{E} {}^{F}_{G}$,

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• Modulation

21/10/2025

→ We re-consider the directed graph representation where common triads are shown as merged nodes. This flow graph can guide the definition of smooth modulations. Again, we use the three keys **C maj.**, **G maj.** and **D maj.**:



In this flow graph, one sees the following behaviors:

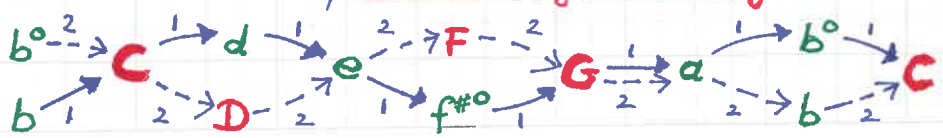
- 1 edge leaves a node: e.g., 1 edge leaves $c^{\#\circ}$, d, b°.
- 1 edge enters a node: e.g., 1 edge enters d, a.
- 2 edges leave a node: e.g., 2 edges leave C, G, b.
- 2 edges enter a node: e.g., 2 edges enter D, e, C.
- 3 edges leave a node: e.
- 3 edges enter a node: G.

One could also use the terms "splitting" and "merging":

- C splits into d and D. ⇒ C can transition to d or D.
- e splits into F, $f^{\#\circ}$ and $f^{\#}$. ⇒ e can transition to F, $f^{\#\circ}$ or $f^{\#}$.
- d and D merge in e. ⇒ Both d and D transition to e.
- F, $f^{\#\circ}$ and $f^{\#}$ merge in G. ⇒ F, $f^{\#\circ}$ and $f^{\#}$ transition to G.

It is now possible to identify and select various paths from a start key to a new key. For example, one can use:

- From C maj. to G maj.: C - d - e - $f^{\#}$ - G.
- From C maj. to D maj.: C - d - e - $f^{\#}$ - G - a - b - $c^{\#\circ}$ - D.
- Possibly from D maj. to C maj.: D - e - $f^{\#\circ}$ - G - a - b° - C.



→ The left figure shows a potential "meandering": C-G-C-G-C...

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→ Considering modulation C maj. ↔ G maj., the pivot chords serving as "modulation portals" are e, G, a and C. In the immediate key-inherent "triad neighborhoods," one sees:

- Pivot e: D F A (1) → E G B (2) → F A C; D F# A (2) → E G B (1) → F# A C
  1 Cmaj. ↦ Gmaj.
  2 Gmaj. ↦ Cmaj.

- Pivot(s) G-a: F A C (1) → G B D (1/2) → A C E (2) → B D F; F# A C (2) → G B D (1/2) → A C E (1) → B D F#
  1 Cmaj. ↦ Gmaj.
  2 Gmaj. ↦ Cmaj.

- Pivot C: B D F (1) → C E G (2) → D F A; B D F# (2) → C E G (1) → D F# A
  1 Cmaj. ↦ Gmaj.
  2 Gmaj. ↦ Cmaj.

These "portal paths" provide ideas for potential modulations.

→ Note. It is possible to define 9th chords that "contain as sub-chords" (triads) key-inherent chords (triads) of the two keys considered for modulation. For example, C E G B D = C E G ∪ G B D or F A C E G = F A C ∪ C E G; thus, it should be possible to consider these 9th chords for potential modulations C maj. ↔ G maj. and F maj. ↔ C maj., respectively.

→ $G^b\ D^b\ A^b\ E^b\ B^b\ F\ C\ G\ D\ A\ E\ B\ F^\#$
$e^b\ b^b\ f\ c\ g\ d\ a\ e\ b\ f^\#\ c^\#\ g^\#\ d^\#$
Of course, one can define a multitude of modulation sequences that involve all 12 major and 12 minor keys/scales. For example, one can consider the sequence C e G b D $f^\#$ A $c^\#$ E $g^\#$ B $d^\#$ (e^b) $F^\#$ (G^b) b^b D^b f A^b c E^b g B^b d F a C – shown in the figure. Each step in this sequence requires one to add/take away only one accidental (#, b) at most – since the sequence performs modulation steps according to the circle-of-fifths order.

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→ It is also interesting to consider the progression of all (major) triads, following the circle-of-fifths order:

CEG DF#A EG#B F#A#C# G#B#D# A#C##E# B#D##F##(CEG)
GBD AC#E BD#F# C#E#G# D#F##A# E#G##B#

This sequence of triads can inform smooth modulations.

→ We briefly consider the "circle-of-fifths neighborhood" F-d-C-a-G-e to show pivot chords/triads in the context of both major and minor keys. As before, keys share major and minor triads that can play a role when "pivoting."

key	C	C_D	D	D_E	E	F	F_G	G	G_A	A	A_B	B	C
F	V		vi		vii°	I		ii		iii	IV		V
d	VII	vii°	i		ii°	III		iv		v	VI		VII
C	I		ii		iii	IV		V		vi		vii°	I
a	III		iv		v	VI		VII	vii°	i		ii°	III
G	IV		V		vi		vii°	I		ii		iii	IV
e	VI		VII	vii°	i		ii°	III		iv		v	VI

This table shows that the six keys F, C, G and d, a, e share the triad CEG, and also have the triad ACE in common. "Unconventionally," the table uses the notation "vii°" to indicate that one can raise the seventh note of the natural, relative minor key/scale — to produce the harmonic minor key/scale and introduce a lead note. In the three minor keys included in the table, this change means: d min.: VII = CEG → C#EG = vii°; a min.: VII = GBD → G#BD = vii°; e min.: VII = DF#A → D#F#A = vii°. Considering these three minor keys/scales in more detail, these are all triad changes when moving from relative to harmonic minor: d min.: FAC, ACE, CEG → FAC#, AC#E, C#EG; a min.: CEG, EGB, GBD → CEG#, EG#B, G#BD; e min.: GBD, BDF#, DF#A → GBD#, BD#F#, D#F#A.

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➔ The modulation from the key of C maj. to the key of G maj. is described in more detail in the following – using a pivot.

F maj.: FAC (FI)

C maj.: FAC (CII)

G maj.:

D maj.:

PIVOTS	
CEG (CI)	GBD (CV)
CEG (GIV)	GBD (GI)

DF#A (GV)

DF#A (DI)

First, one can consider a full, authentic cadence for the C maj. key, e.g., CI – CIV – CV – CI = CEG – FAC – GBD – CEG.

Second, one can consider a full, authentic cadence for the G maj. key, e.g., GI – GIV – GV – GI = GBD – CEG – DF#A – GBD.

Thus, when one views these two cadences in a "combined" form, one moves effectively from CI to GI – where the triads CEG (CI, GIV) and GBD (CV, GI) serve as pivot triads to "blend" C maj. and G maj. smoothly. The triad GV = DF#A introduces the needed accidental note F#.

Further, one must also keep in mind that the VOICING of the triads used in the "modulation cadences" should move notes minimally when moving from triad to triad. Therefore, one must use root position and 1st and 2nd inversions of all triads. Superscripts are used to indicate these 3 positions, e.g., $CI^0 = CEG$, $CI^1 = EGC$, $CI^2 = GCE$.

For example, the considered cadences can be voiced more smoothly by using the following triad sequences:

C maj. cadence: $CI^1 - CIV^0 - CV^0 - CI^2 = EGC - FAC - GBD - GCE$;

G maj. cadence: $GI^0 - GIV^2 - GV^2 - GI^1 = GBD - GCE - ADF^{\#} - BDG$.

When viewing the combined sequence of these 8 triads, one sees that notes move up/down by 0, 1 or 2 semi-tones only.

Instead of this pair of full, authentic cadences, one can also consider the use of the shorter authentic cadences:

C maj. cadence: $CI^1 - CV^0 - CI^2 = EGC - GBD - GCE$;

G maj. cadence: $GI^0 - GV^2 - GI^1 = GBD - ADF^{\#} - BDG$.

This triad progression can be improved, as far as voicing is concerned, by using the following triads:

C maj. cadence: $CI^1 - CV^2 - CI^1 = EGC - DGB - EGC$;

G maj. cadence: $GI^2 - GV^0 - GI^2 = DGB - DF^{\#}A - DGB$.

This second progression of 6 triads ensures that triad notes move up/down by only 0, 1 or 2 semi-tones.

→ One can repeatedly use a chosen, smooth modulation, to move from a key to another key to another key etc. We use the "smooth-voicing" cadence for I-IV-V-I as an example:

C maj. cad.: $CI^1 - CIV^0 - CV^0 - CI^2 = EGC - FAC - GBD - GCE$;

G maj. cad.: $GI^0 - GIV^2 - GV^2 - GI^1 = GBD - GCE - ADF^{\#} - BDG$;

G maj. cad.: $GI^1 - GIV^0 - GV^0 - GI^2 = BDG - CEG - DF^{\#}A - DGB$;

D maj. cad.: $DI^0 - DIV^2 - DV^2 - DI^1 = DF^{\#}A - DGB - EAC^{\#} - F^{\#}AD$;

D maj. cad.: $DI^1 - DIV^0 - DV^0 - DI^2 = F^{\#}AD - GBD - AC^{\#}E - ADF^{\#}$;

A maj. cad.: $AI^0 - AIV^2 - AV^2 - AI^1 = AC^{\#}E - ADF^{\#} - BEG^{\#} - C^{\#}EA$.

Only three 6-triad pro-gression

This example represents a smooth overall modulation from C maj. to A maj. keys/scales. Effectively, one moves from the triad EGC to triad $C^{\#}EA$ – via the sequence $EGC - BDG - F^{\#}AD - C^{\#}EA$. The introduced accidentals $F^{\#}$, $C^{\#}$ and $G^{\#}$ are "double-underlined" in the triads that introduce them in the modulation. The entire sequence of triads (24!) included in the "modulation sequences" provided follows the principle of smoothest-possible voicing. Ultimately, one can modulate through the entire circle of fifths.

→ The full, authentic cadences one can use for a smooth modulation can be "reduced" to these essential triads:

C maj. cad.: $CI^1 - CIV^\circ - \boxed{CV^\circ - CI^2}$

G maj. cad.: $\boxed{GI^\circ - GIV^2} - GV^2 - GI^1$

≙ EGC - FAC - GBD - GCE - ADF# - BDG

This example for the modulation Cmaj. to Gmaj. can be repeated:

	Modulation	Triad 1	2	3	4	5	Triad 6
1	C→G	EGC	FAC	GBD	GCE	ADF#	BDG
2	G→D	BDG	CEG	D F# A	DGB	EAC#	F# A D
3	D→A	F# A D	GBD	AC# E	AD F#	BEG#	C# E A
4	A→E	C# E A	D F# A	EG# B	EAC#	F# B D#	G# B E
5	E→B	G# B E	AC# E	B D# F#	BEG#	C# F# A#	D# F# B
6	B→F#	D# F# B	EG# B	F# A# C#	F# B D#	G# C# E#	A# C# F#
7	F#→C#	A# C# F#	B D# F#	C# E# G#	C# F# A#	D# G# B#	E# G# C#
8	C#→G#	E# G# C#	F# A# C#	G# B# D#	G# C# E#	A# D# F##	B# D# G#
9	G#→D#	B# D# G#	C# E# G#	D# F## A#	D# G# B#	E# A# C##	F## A# D#
10	D#→A#	F## A# D#	G# B# D#	A# C## E#	A# D# F##	B# E# G##	C## E# A#
11	A#→E#	C## E# A#	D# F## A#	E# G## B#	E# A# C##	F## B# D##	G## B# E#
12	E#→B#	G## B# E#	A# C## E#	B# D## F##	B# E# G##	C## F## A##	D## F## B# = EGC
7	Gb→Db	Bb Db Gb	Cb Eb Gb	Db F Ab	Db Gb Bb	Eb Ab C	F Ab Db
8	Db→Ab	F Ab Db	Gb Bb Db	Ab C Eb	Ab Db F	Bb Eb G	C Eb Ab
9	Ab→Eb	C Eb Ab	Db F Ab	Eb G Bb	Eb Ab C	F Bb D	G Bb Eb
10	Eb→Bb	G Bb F	Ab C Eb	Bb D F	Bb Eb G	C F A	D F Bb
11	Bb→F	D F Bb	Eb G Bb	F A C	F Bb D	G C E	A C F
12	F→C	A C F	Bb D F	C E G	C F A	D G B	E G C

In each row, the "start" key is shown in the column "Triad 1"; the "goal" key is shown in the column "Triad 6." The last, 3rd note of "Triad 1" and "Triad 6" define the "start" and "goal" keys. Performing all modulations, from 1 to 12, represents a "modulation circle" that moves through all major keys in the sequence of the circle of fifths. ONE CAN MOVE FORWARD OR BACKWARD

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• Modulation

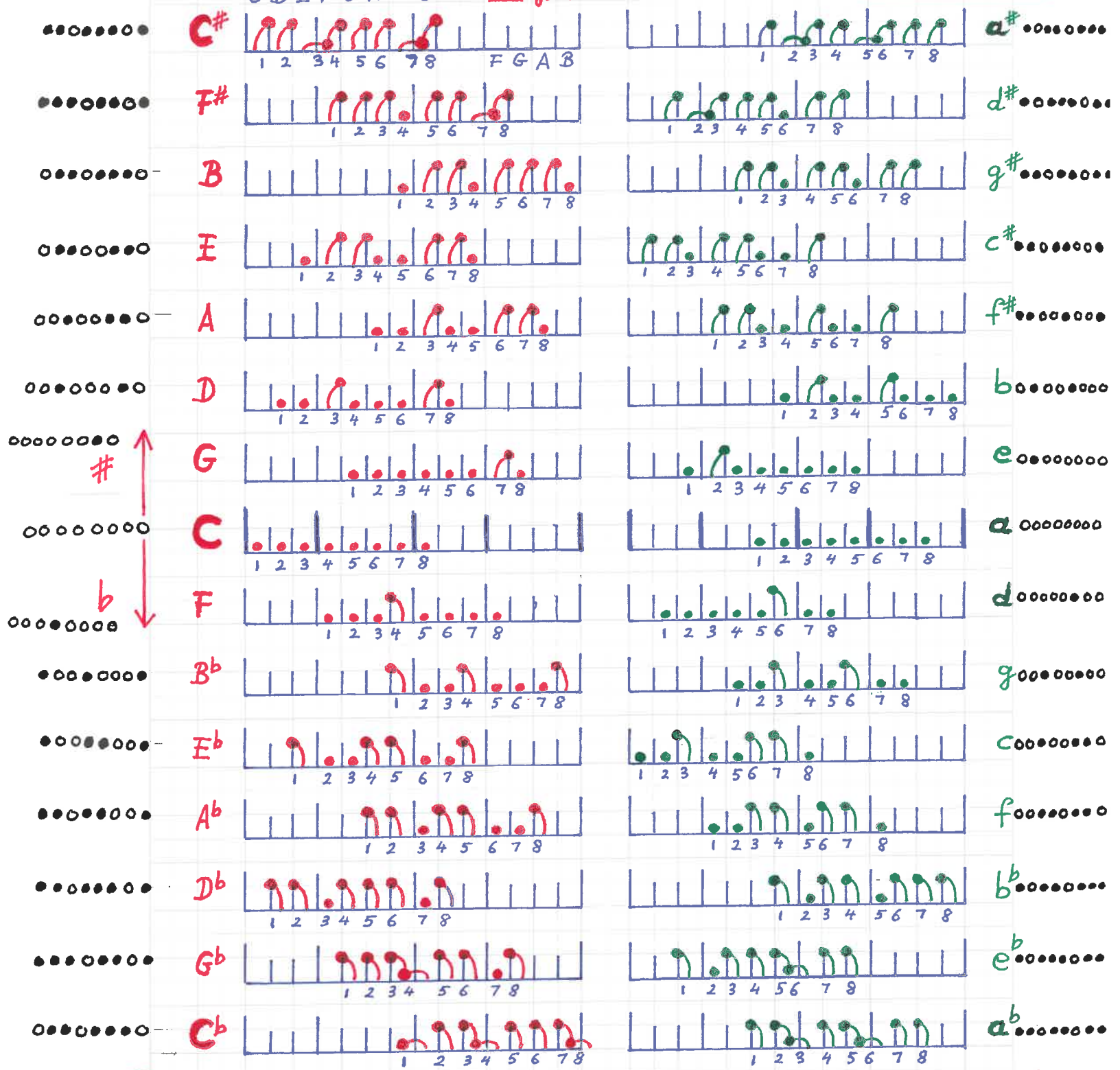
2/13/2025

→ In order to perform modulations on the keyboard with all 24 major and minor keys, "effective, efficient memorization" is needed.

Key color sequence:

C D E F G A B C D E MAJOR

MINOR (relative)



1) Memorize #'s and b's of all keys.

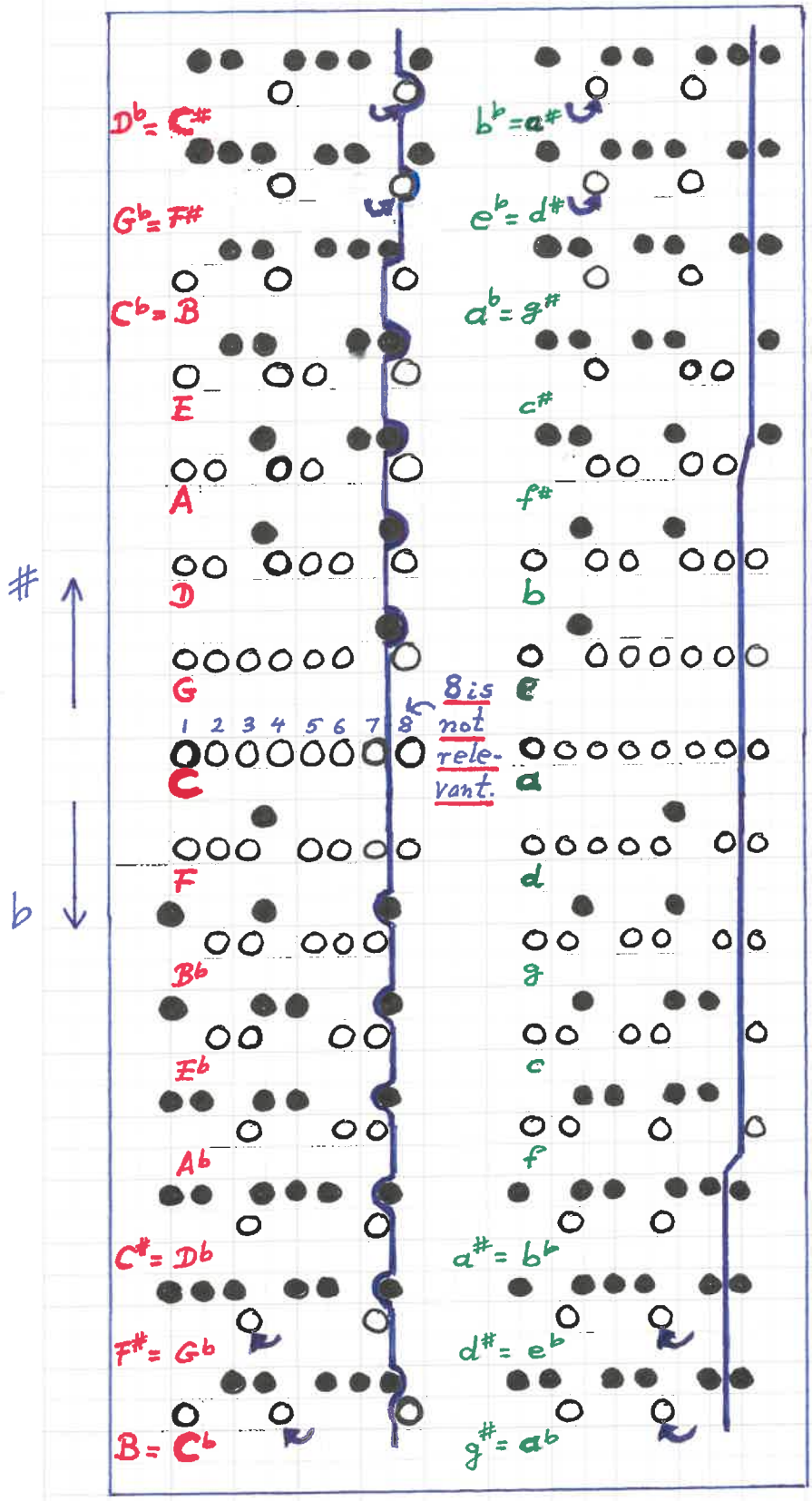
2) Memorize locations of all #'s, b's.

3) Number octave notes 1, 2, 3, ..., 8.

4) #'s: move "up-right"; b's: move "up-left."

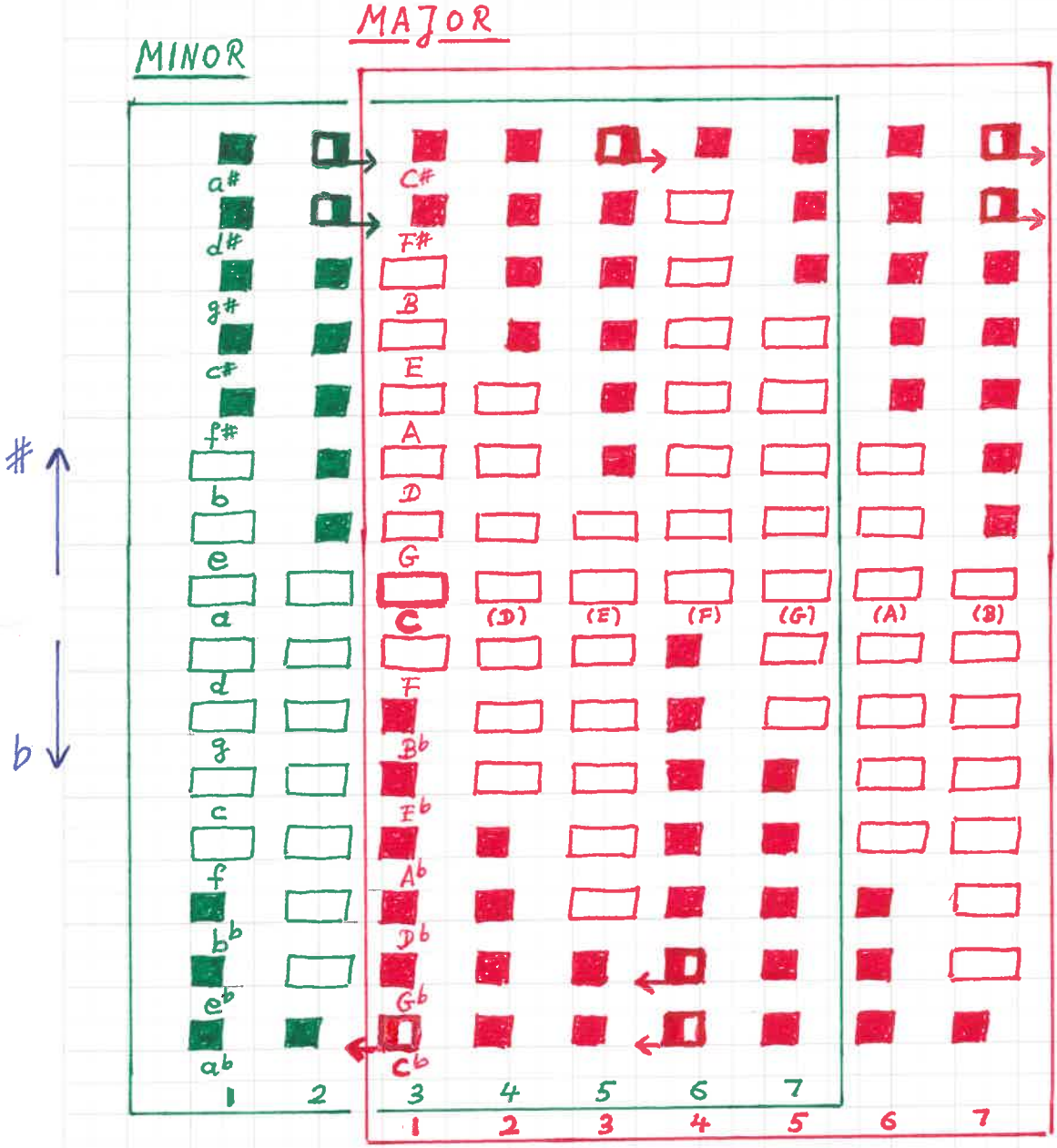
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→ One can also memorize the "shapes and patterns" of the keys (on the keyboard) of the 24 scales by "extracting" the 8 white and black keys that constitute one octave of a specific key/scale. The left table shows the "shapes and patterns" of all 12 major keys (left column) and all 12 relative minor keys. The order of the keys is using the circle-of-fifths order, to emphasize and "see" the gradual changes of key "patterns and shapes" when '#s' and 'bs' are added in a one-by-one fashion. Therefore, one can also focus on and memorize how a "pattern and shape" changes when "the next '#' ('b')" is introduced. The 8th key/note is not "pattern-relevant."



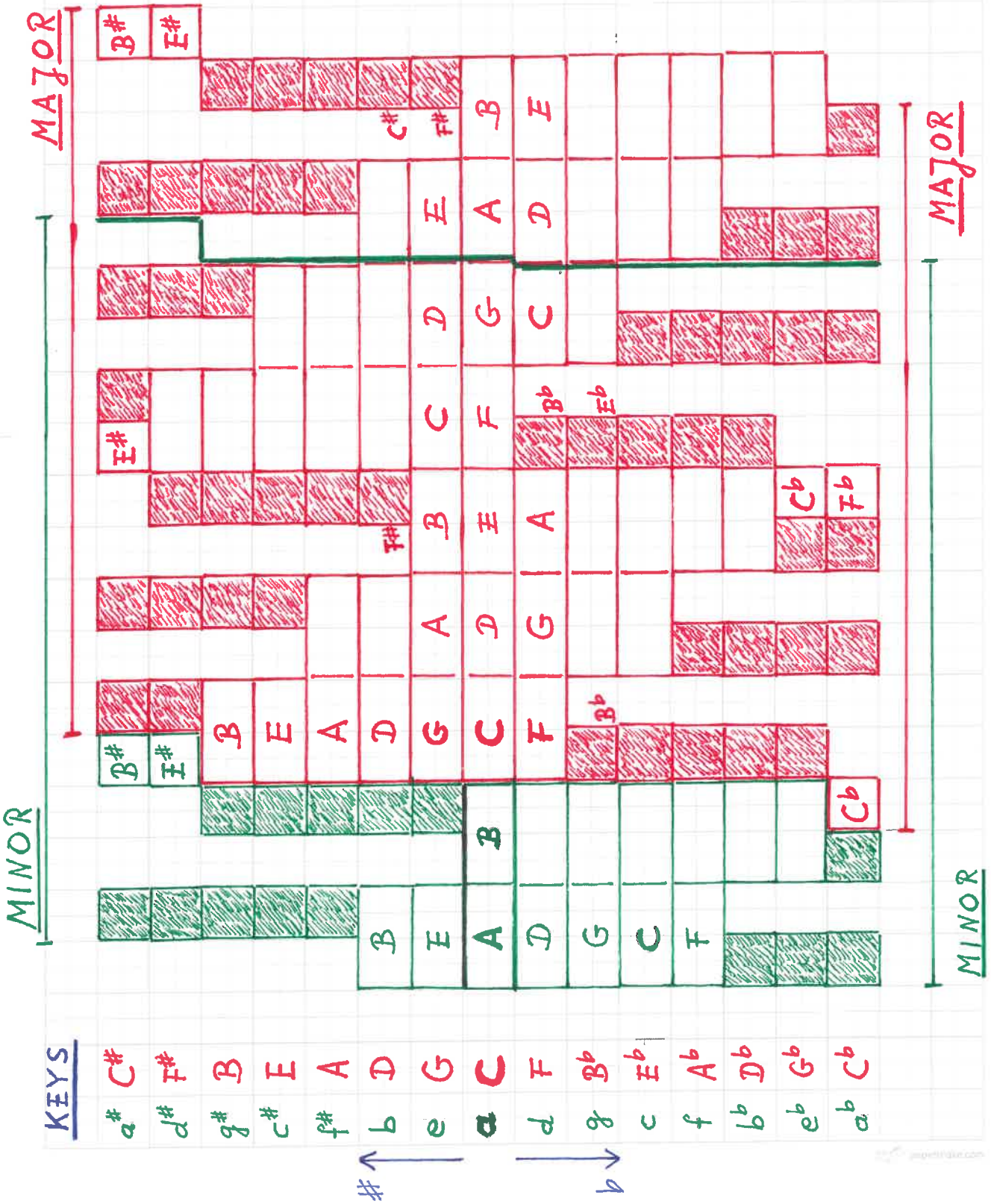
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→ The figure below is a compact representation of all 12 major and 12 relative minor keys/scales – i.e., a representation of the keys on the keyboard to be played.



The figure shows that B, F#, C# (g#, d#, a#) and C♭, G♭, D♭ (a♭, e♭, b♭), respectively, are indeed "not equal" – concerning their harmonic functions.

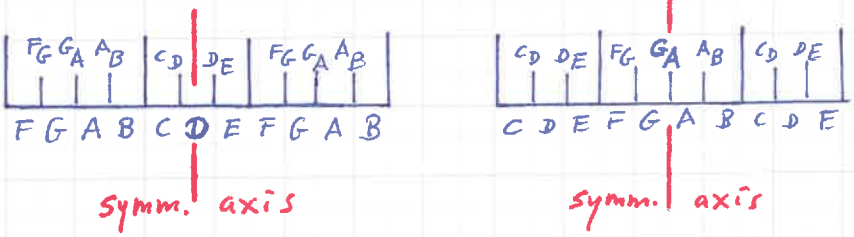
The visualization on this page shows another possibility to represent the notes and corresponding keys on the keyboard of all 12 major and associated 12 relative minor keys/scales. Again major and minor scales are overlaid.



• Mathematical Transformations & Effects

2/17/2025

1) Keyboard Symmetry



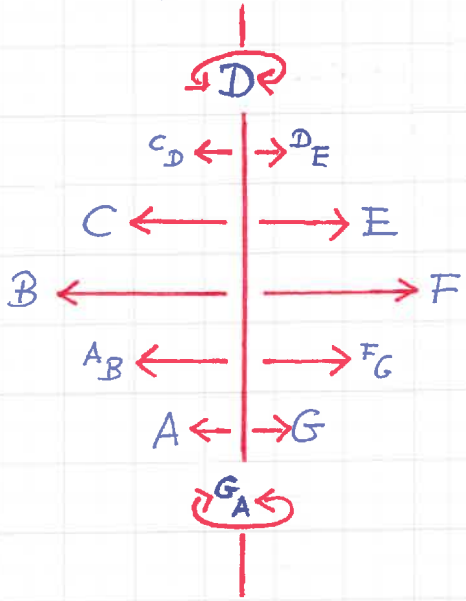
• Symmetry axis at notes D and G♯

• The two symmetry axes produce the same note mappings:

C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
E	D♯	D	C♯	C	B	A♯	A	G♯	G	F♯	F

mirror images / reflections

• The "cyclic keyboard" makes it possible to use a circle:



• The left representation of "axis symmetry" shows the note pairs of notes that are "mirror images" of each other.

• It is interesting to understand how this mapping affect chords. For example, one can mirror the key-inherent triads of C major:

$C^1 E^2 G^3 \mapsto A^{3'} C^{2'} E^{1'}$ $\quad I \mapsto vi$

$D^1 F^2 A^3 \mapsto G^{3'} B^{2'} D^{1'}$ $\quad ii \mapsto V$

$E^1 G^2 B^3 \mapsto F^{3'} A^{2'} C^{1'}$ $\quad iii \mapsto IV$

$F^1 A^2 C^3 \mapsto E^{3'} G^{2'} B^{1'}$ $\quad IV \mapsto iii$

$G^1 B^2 D^3 \mapsto D^{3'} F^{2'} A^{1'}$ $\quad V \mapsto ii$

$A^1 C^2 E^3 \mapsto C^{3'} E^{2'} G^{1'}$ $\quad vi \mapsto I$

$B^1 D^2 F^3 \mapsto B^{3'} D^{2'} F^{1'}$ $\quad vii^o \mapsto vii^o$

C maj. triads mapped to other C maj. triads !

• A chord $N_1 N_2 \ldots N_n$ is "mirrored" to the chord $N_n' N_{n-1}' \ldots N_1'$ – where N_i' is the mirror image of note N_i. Example:

$C E G = N_1 N_2 N_3 \longmapsto$

$N_3' N_2' N_1' = G' E' C' = A C E$.

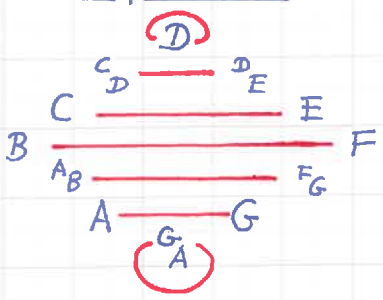
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1) TRANSLATION

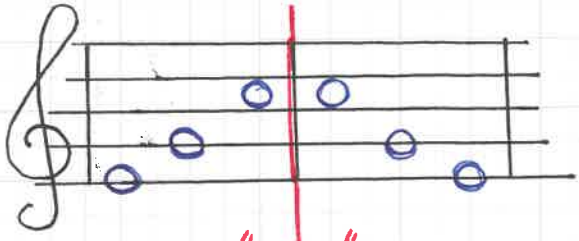
- "spatial": transposing a note/chord/key "up or down" – e.g., modulation to another key
- "temporal": voices (soprano, alto, tenor, bass) are "offset"/"delayed" relative to each other (canon, fugue)

2) REFLECTION

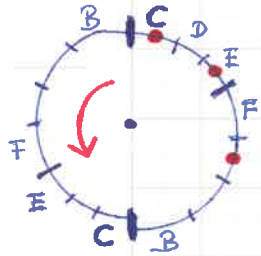
- "spatial"



- "temporal"

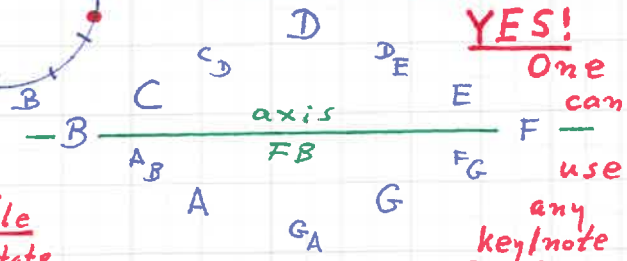


ROTATION:



View the keyboard as a circle ⇒ can "rotate notes/chords"

- Besides the axis DG_A, are other axes possible? ⟺ For example, if one could view the line FB (left figure) as an "allowable axis", one could concatenate two reflections:



YES! One can use any key/note as reflection axis.

i) $C^1 E^2 G^3 \xrightarrow[D G_A]{\text{axis}} A^{3'} C^{2'} E^{1'} \xrightarrow[FB]{\text{axis}} F_G\, A_B\, C_D$?

⇒ ii) $Ref_{FB}(Ref_{DG_A}(CEG)) = F_G A_B C_D$.

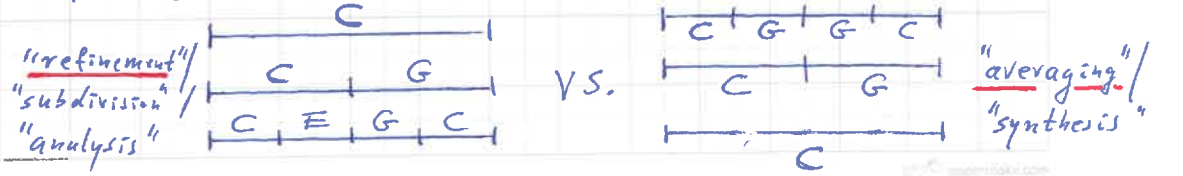
⇒ Possibility to define "ROTATION" VIA TWO REFLECTIONS?

3) SCALING – w.r.t. a scaling CENTER

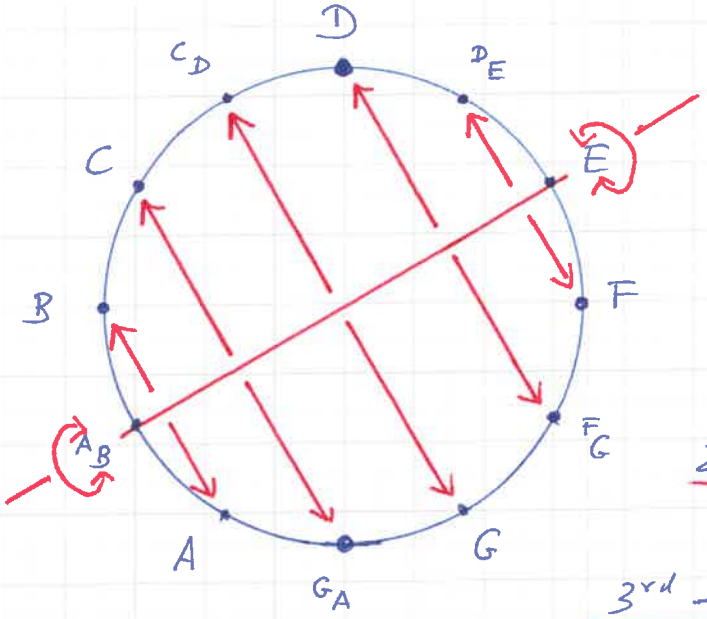
- "spatial": e.g., scale intervals relative to a "scaling center":

(center = C:) $C \overset{4}{\underset{4}{\rightarrow}} E \overset{7}{\underset{3}{\rightarrow}} G$ $\xrightarrow[\text{by } 2]{\text{scale}}$ $C \overset{8}{\underset{8}{\rightarrow}} G_A \overset{14}{\underset{6}{\rightarrow}} D$ ⇐ Sounds "good"?

- "temporal": similar to "multi-resolution" methods – longer & shorter notes:



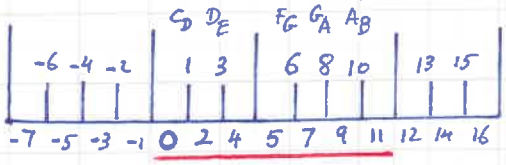
4) REFLECTION w.r.t. any reflection axis



- A general reflection can be viewed as a "geometrical concatenation":
  1st – "translate" to a default center of reflection / standard axis of reflection;
  2nd – "reflect" relative to this default standard axis;
  3rd – "translate" back.

⇒ Example: [see figure above]: reflect relative to axis E^{A_B}
⇒ note C is reflected to G_A.

- Performing this general reflection with note "indices":



⇐ This figure shows a possible indexing / numbering of keys / notes. C ≙ 0 is used as default center / axis of reflection.

⇒ Example: "Reflect F (≙ 5) w.r.t. axis F_G (≙ 6)
1st – "translate by -6" ⇒ F ↦ B (≙ -1)
2nd – "reflect w.r.t. axis C (≙ 0)": B ↦ C_D (≙ 1)
3rd – "translate by +6" ⇒ C_D ↦ G (≙ 7)
⇒ F ⟼ G ✓

- Considering the above figure again, we can reflect triads that are key-inherent to C major relative to the E^{A_B} axis:

C_{maj} {
I $C^1 E^2 G^3 \mapsto {}^{3'}C_D\ E\ {}^{2'}G_A$ = E vi
ii $D\ F\ A \mapsto B\ D_E\ F_G$ = E V
iii $E\ G\ B \mapsto A\ C_D\ E$ = E IV
IV $F\ A\ C \mapsto G_A\ B\ D_E$ = E iii
}
C_{maj} {
V $G\ B\ D \mapsto F_G\ A\ C_D$ = E ii
vi $A\ C\ E \mapsto E\ G_A\ B$ = E I
vii° $B\ D\ F \mapsto D_E\ F_G\ A$ = E vii°
}

⇒ C maj. triads to E maj. triads!

- Modulation

3/1/2025

Similarity of Seventh Chords and Keys/Scales Defined by the Numbers of Chord Notes Belonging to Specific Keys/Scales

7th Chord	C-E-G-B	C-E-G-Bb	C-Eb-G-Bb	C-Eb-Gb-Bb	C-Eb-Gb-A
F# [d#]	*	*	* *	***	* *
B [g#]	* *	* *	* *	***	**
E [c#]	* *	*	*	**	**
A [f#]	* *	*		*	**
D [b]	***	**	*	*	**
G [e]	****	***	* *	* *	* **
C [a]	****	***	* *	*	* *
F [d]	***	****	* **	* *	* *
Bb [g]	* *	* **	****	** *	** *
Eb [c]	* *	* **	****	** *	**
Ab [f]	* *	* **	****	** *	**
Db [bb]	*	* *	** *	****	***

Note. The numbers of chord notes belonging to a specific key/scale are equal to the numbers of '*' (0, 1, 2, 4). For example, all 4 notes of 7th chord C-E-G-Bb belong only to F maj. (d min.). Thus, this C7 chord can be used as a "pivot chord" to modulate from C maj. to F maj.: C-E-G → C-E-G-Bb → C-F-A.